SMT359

Additional Exercises

Contents

Exercises for Book 1

Book 1 Chapter 1

Exercise 1.1 What fraction of their electrons would a person need to lose to have a net charge of $1\,\mu$C? For simplicity, imagine that the person has a mass of 60 kg and is composed entirely of water molecules. Each water molecule has a mass of 3.0×10^{-26} kg and contains 10 electrons.

Exercise 1.2 Suppose that one gram of protons is separated from another gram of protons by 3.84×10^5 km (the distance between the Earth and the Moon). Estimate the magnitude of the electrostatic force between these two concentrations of charge.

Exercise 1.3 Two identical point charges, Q, are a distance d apart. A point charge q is placed on the line joining these charges, midway between them. What value must be chosen for q to ensure that all three charges experience zero electrostatic force? Is the system then in *stable* equilibrium?

Exercise 1.4 Four identical point charges, q, are at the corners of a square with length of side d. (a) What is the magnitude of the electric field at the centre of the square? (b) What is the magnitude of the electric field at the centre of the square when one of the corner charges is removed?

Exercise 1.5 Four charges are arranged in a square with side of length d (see Figure 1). The (x, y) coordinates of the charges are as follows: $4q$ at $(0, d)$, $2q$ at (d, d), $2q$ at $(0, 0)$ and $-q$ at $(d, 0)$. What is the electric field at the centre of the square?

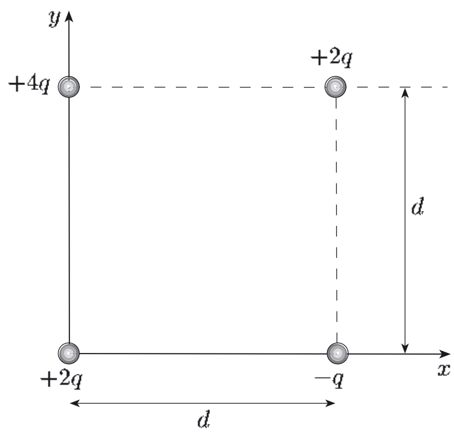

Figure 1 For Exercise 1.5.

Exercise 1.6 What is the electric field at the centre of a uniformly charged disk of radius R and charge Q?

Exercise 1.7 A positive charge Q is uniformly distributed over a thin circular ring of radius R. (a) What is the electric field at the centre of the ring? (b) What is the electric field at the centre of the ring if a short segment of charge of length $l \ll R$ is removed from the ring?

Exercise 1.8 An electric field vector $\mathbf{E}$ has the value $(3\mathbf{e}_x + 4\mathbf{e}_y)\,\mathrm{N\,C^{-1}}$. What is the magnitude of this field and what is its unit vector? Find the component of the field in the direction of the unit vector $\widehat{\mathbf{u}} = (\mathbf{e}_x - \mathbf{e}_y)/\sqrt{2}$.

Book 1 Chapter 2

Exercise 2.1 A charge distribution is cylindrically symmetrical around the z-axis and is described by the charge density

$$\rho(r) = \frac{A}{r} \quad \text{for } 0 \le z \le L \text{ and } 0 < r \le R,$$

where r is the distance from the z-axis and A is a constant. This charge distribution is independent of z for $0 \le z \le L$, but vanishes outside this range and for $r > R$. What is its total charge?

Exercise 2.2 A hydrogen atom consists of a proton, represented by a point charge $+e$ at the origin, and an electron, represented by a spherically symmetric charge distribution of charge density

$$\rho(r) = A \exp(-2r/a_0),$$

where r is the distance from the origin, A is a constant and $a_0 = 5.29 \times 10^{-11}$ m is the Bohr radius. (a) Use the fact that a hydrogen atom is electrically neutral to find a formula for A. (b) Hence evaluate the charge density of the electron cloud at the Bohr radius, a_0. You may use the standard integral

$$\int_0^\infty x^2 \exp(-x)\,\mathrm{d}x = 2.$$

Exercise 2.3 Consider the electric field $\mathbf{E} = A\exp(-r/a)\,\mathbf{e}_r$, where $\mathbf{e}_r$ is a radial unit vector pointing away from the origin and A and a are positive constants. Show that such a field could not be produced by an isolated point charge at the origin, even one moving rapidly enough to rule out any arguments based on Coulomb's law.

Exercise 2.4 Consider a spherically symmetric vector field $\mathbf{F} = \exp(-\lambda r)\,\mathbf{e}_r$, where λ is a positive constant and $\mathbf{e}_r$ is a radial unit vector pointing away from the origin. Does $\mathbf{F}$ obey the divergence theorem? Is the flux of $\mathbf{F}$ equal to zero through a closed surface that does not enclose the origin?

Exercise 2.5 Show that there is no electric field inside a uniformly charged infinitely long cylindrical shell.

Exercise 2.6 Consider a cylinder-shaped uniform distribution of positive charge. The cylinder has finite length and negligible radius. The electric field due to this distribution is measured at a point P *outside* the charge distribution and on its central axis. If this charge distribution were shrunk down to its central point, would the magnitude of the field at P increase or decrease?

Exercise 2.7 The electric flux over a spherical surface of radius R, centred on the origin, does not depend on any charges that are outside this surface. Does this mean that the electric field at any point a distance R from the origin does not depend on charges that are further from the origin than R?

Exercise 2.8 Is it possible to directly use the integral version of Gauss's law to find the electric field at every point in space due to a cube of charge with length of side L and a uniform charge density ρ?

Exercise 2.9 The membrane of a spherical animal cell has a uniform surface charge density of $-2.5 \times 10^{-6}\,\mathrm{C\,m^{-2}}$ on its inside surface and $+2.5 \times 10^{-6}\,\mathrm{C\,m^{-2}}$ on its outside surface. The thickness of the membrane is much less than the radius of the cell. Ignoring any polarization effects and assuming there are no other concentrations of charge, determine the magnitude of the electric field (a) inside the membrane, (b) inside the cell and (c) outside the cell.

Exercise 2.10 An isolated infinite slab of uniform charge density $\rho > 0$ lies between $z = -d/2$ and $z = d/2$. (a) Use the integral version of Gauss's law to find the magnitude of the electric field at all points inside and outside the slab. (b) Confirm that your answer is consistent with the differential version of Gauss's law.

Exercise 2.11 An infinite plane at $z = d/2$ has a uniform surface charge density σ (see Figure 2). It is in contact with an infinite slab of charge with uniform charge density ρ extending from $z = -d/2$ to $z = d/2$. All the charges are fixed. Where, if anywhere, in the slab does the electric field vanish?

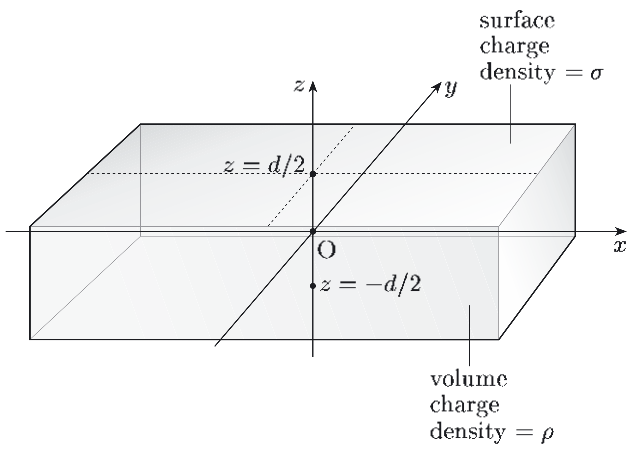

Figure 2 For Exercise 2.11.

Exercise 2.12 A hydrogen atom can be modelled as a point charge $+e$ at the centre of a spherically symmetric cloud of negative charge with charge density $\rho = -e\exp(-2r/a_0)/(\pi a_0^3)$, where $a_0 = 5.29 \times 10^{-11}$ m is the Bohr radius. What is the radial component of the electric field at a_0?

You may use the fact that $\int_0^2 x^2 \exp(-x)\,\mathrm{d}x = 0.647$.

Exercise 2.13 A hemispherical shell with radius R has a uniform areal charge density σ (see Figure 3). The centre of the hemisphere is at the origin, and the shell extends out towards values with $z > 0$. Consider now the imaginary surface S that forms the base of the shell, i.e. the circular, flat surface with radius R in

the xy-plane. Show that, at all points on the surface S, the electric field is perpendicular to S.

Exercise 2.14 (a) A point charge Q is at the centre of a cube. What is the electric flux over each face of the cube? (b) The charge is now moved far into a corner of the cube, whilst remaining wholly within the cube. What is the electric flux over each face of the cube in this situation? *Hint: Embed the original cube in a larger cube with double the side length so that the point charge is close to the centre of this larger cube.*

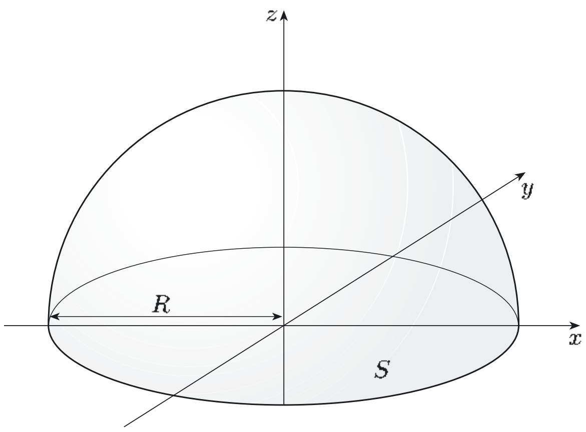

Figure 3 For Exercise 2.13.

Exercise 2.15 Describe the electric field lines outside two concentric spherical conducting shells. The inner shell has a positive charge $q_1 > 0$ and the outer shell has a negative charge $q_2 < 0$. Consider the three cases $q_1 + q_2 < 0$, $q_1 + q_2 = 0$ and $q_1 + q_2 > 0$.

Exercise 2.16 The detailed study of ions is a topic of current interest in physics and chemistry. These studies are aided by building ion traps, which confine the ions in a trapping region which is a vacuum occupied only by trapped ions themselves. Consider a proposal for building a trap by distributing charges outside the trapping region in such a way that they produce a suitable electric field to trap positive ions. To achieve stable equilibrium, the electric field would have to exert a restoring force on the trapped ions, always pushing them back inside the trapping region. Use Gauss's law to show that this proposal is unworkable; the required electric field cannot be created by charges outside the trapping region.

Exercise 2.17 An electric field that points in the x-direction depends only on x and is equal to zero at $x = 0$. The charge density responsible for this field is $\rho(x) = \rho_0 \cos(kx)$. What is the electric field?

Book 1 Chapter 3

Exercise 3.1 The Earth's magnetic field is similar to that of a magnetic dipole with a magnetic moment of 8.0×10^{22} A m^2. If this were due to a loop of current with radius 2350 km, what would the magnitude of current be?

Exercise 3.2 The current density in a region of space varies as

$$\mathbf{J} = \mathbf{A} \exp(-\mathbf{k} \cdot \mathbf{r}),$$

where $\mathbf{A}$ and $\mathbf{k}$ are constant non-zero vectors. What is the direction of the current flow? Under what conditions does this current density lead to no accumulation of charge?

Exercise 3.3 A circular loop of wire carries a steady current. Use the Biot–Savart law to show that the magnetic force on any segment of the wire is in the plane of the loop and points radially outwards from its centre.

Exercise 3.4 Show that the electric field in an empty Penning trap is bound to have a radial component that opposes the confining effect of the magnetic field. (You will need to think back to an earlier chapter to answer this question, remembering that the trapping region is a vacuum with zero charge.)

Book 1 Chapter 4

Exercise 4.1 A current distribution is spherically symmetric, flowing radially away from the origin. Use symmetry principles to show that the magnetic field of this current distribution is radial. Then use the no-monopole equation to show that the magnetic field actually vanishes.

Exercise 4.2 A thick circular ring of current is centred on the origin and lies in the plane $z = 0$. Use symmetry principles to express the magnetic field near this current distribution in the simplest possible terms.

Exercise 4.3 A long coaxial cable consists of a cylinder surrounded by a cylindrical tube, both with the same central axis. The cylinder and the tube are both conductors, but the space between them is filled with an insulator. Steady uniform currents flow in the cylinder and the tube. These currents have the same magnitude I, but flow in opposite directions. What is the magnetic field outside the coaxial cable?

Exercise 4.4 Two infinite current sheets are arranged so that they are parallel, and separated by a distance d. Sheet 1 is in the plane defined by $z = -d/2$, sheet 2 is in the plane defined by $z = +d/2$. Each sheet is made up of wires aligned in the x-direction with N/l wires per unit length in the y-direction. Each wire carries a current of magnitude I, but one sheet carries it in the positive x-direction while the other carries it in the negative x-direction. What is the magnetic field between the current sheets?

Book 1 Chapter 5

Exercise 5.1 Calculate the potential energy of a uniformly charged sphere of charge Q and radius R.

Exercise 5.2 What is the potential at the centre of a uniformly charged sphere of charge Q and radius R?

Exercise 5.3 What is the potential at the centre of a uniformly charged disk of charge Q and radius R?

Exercise 5.4 In some respects an atomic nucleus behaves like a uniform spherical distribution of charge. A spherical nucleus of charge Q and radius R divides into two isolated spherical nuclei, each of charge $Q/2$ and radius $R/2^{1/3}$. By how much does the electrostatic potential energy of the system change?

Exercise 5.5 In a particular lightning flash, the potential difference between the cloud and the ground is 1.0×10^9 V and the quantity of charge transferred is 30 C. What is the change in energy of the transferred charge?

Exercise 5.6 Show that, in equilibrium, the electric field is perpendicular to the surface of any conductor.

Book 1 Chapter 6

Exercise 6.1 Consider a rectangular conducting loop ABCD in the plane $z = 0$ moving with constant velocity $\mathbf{v} = v_x \mathbf{e}_x$ in the x-direction (see Figure 4). The points ABCD defining the loop initially have the following (x, y) coordinates: $\mathrm{A}(0, L_1)$, $\mathrm{B}(0, 0)$, $\mathrm{C}(L_2, 0)$, $\mathrm{D}(L_2, L_1)$. Hence sides BA and CD are parallel to the y-axis and have length L_1. Sides AD and BC are parallel to the x-axis and have length L_2. A magnetic field $\mathbf{B} = B_z(x)\,\mathbf{e}_z$ points in the z-direction and is a function of x, but is independent of y, z and t. Show that the induced emf in ABCD is

$$V_{\text{emf}} = v_x L_1 [B_z(x) - B_z(x + L_2)]$$

where x is the instantaneous x-coordinate of side AB and $x + L_2$ is the instantaneous x-coordinate of side DC.

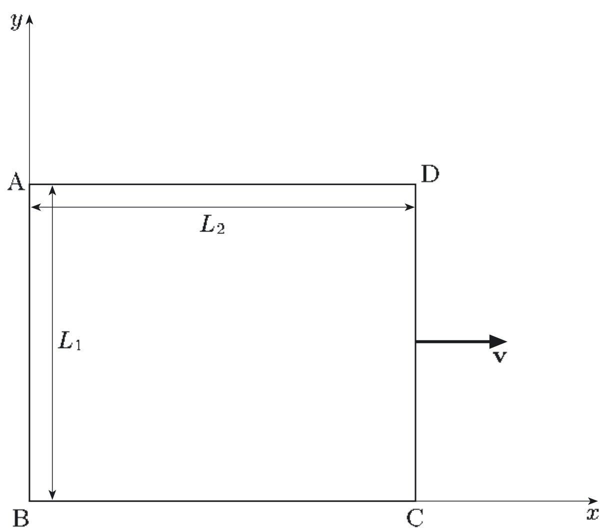

Figure 4 For Exercise 6.1.

Exercise 6.2 A rigid loop maintains a fixed orientation whilst moving through a uniform static magnetic field. Is any emf induced in the loop?

Exercise 6.3 A constant uniform magnetic field of magnitude B points in the positive z-direction. A conducting rod of length L is pivoted at one end and rotates with angular speed ω anticlockwise (as seen from an observer at $z > 0$) in the plane $z = 0$. Use the magnetic force law to calculate the voltage drop from the pivoted to the free end of the rod.

Exercise 6.4 Use Faraday's law to provide an alternative solution to Exercise 6.3.

Book 1 Chapter 7

Exercise 7.1 Inside a conducting medium that obeys Ohm's law, the current density at a given point is related to the electric field at the same point by the equation $\mathbf{J} = \sigma \mathbf{E}$, where σ is the conductivity of the medium. Use this fact, together with the equation of continuity to show that a concentration of excess charge within the medium will decay exponentially in time.

Exercise 7.2 Verify explicitly that the fields

$$\mathbf{E} = f(z - ct)\,\mathbf{e}_x \quad \text{and} \quad \mathbf{B} = \frac{1}{c} f(z - ct)\,\mathbf{e}_y,$$

where $c = 1/\sqrt{\varepsilon_0 \mu_0}$, obey all four of Maxwell's equations in empty space.

Exercise 7.3 Figure 7.10 in Book 1 (p. 182) shows a linearly-polarized monochromatic electromagnetic wave that propagates along the z-axis. Rank the order of the electric field strengths at points P, Q and R, which are all in a plane perpendicular to the direction of propagation and have spatial coordinates $\mathrm{P}(0, 0, 0)$, $\mathrm{Q}(\lambda, 0, 0)$, $\mathrm{R}(\lambda, \lambda, 0)$, where λ is the wavelength.

Exercise 7.4 (a) How would you orient a long straight aerial to detect a plane, linearly-polarized electromagnetic wave?

(b) How would you orient a circular-loop aerial to detect a plane, linearly-polarized electromagnetic wave of wavelength similar to the diameter of the loop?

Exercises for Book 2

Book 2 Chapter 1

Exercise 1.1 Charge is distributed throughout a sphere of radius R with a charge density given by

$$\rho(r) = \begin{cases} kr, & r \leq R, \\ 0, & r > R, \end{cases}$$

where r is the distance from the centre of the sphere and k is a constant.

(a) What is the symmetry of the problem?

(b) Use the integral form of Gauss's law to find the electric field $\mathbf{E}$ at all points in space, both inside and outside the sphere.

(c) Use the differential form of Gauss's law to check your answer to part (b).

Exercise 1.2 Consider the two vector fields

$$\mathbf{F} = x\mathbf{e}_x + (2x - y)\mathbf{e}_y \quad \text{and}$$

$$\mathbf{G} = -2x\mathbf{e}_x + (4y + 5)\mathbf{e}_y.$$

Can either of these vector fields represent a static magnetic field? If appropriate, find the associated current density.

Exercise 1.3 Given the electric field

$$\mathbf{E}(\mathbf{r}, t) = E_0 \sin\left[\omega\left(t - \frac{y}{c}\right)\right] \mathbf{e}_x,$$

where E_0, ω and c are positive constants, use Faraday's law to find the most general form of the associated magnetic field $\mathbf{B}$.

Exercise 1.4 (*The derivation of a wave equation in this question can be done using what you have learned in Book 1. However, this will be discussed in detail in Book 3, and you may prefer to try this question after studying Chapter 1 of Book 3.*)

Use the vector identity

$$\text{curl}\,(\text{curl}\,\mathbf{F}) = \text{grad}\,(\text{div}\,\mathbf{F}) - \nabla^2\mathbf{F}$$

to show that, in a region of empty space that contains no charges or currents, two of Maxwell's equations can be combined to give a wave equation of the form

$$\nabla^2\,\mathbf{E} = \frac{1}{c^2}\,\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

Your answer should establish a relationship between the constant c and two constants appearing in Maxwell's equations.

Exercise 1.5 An electron of mass m and charge $-e$ is moving in a uniform electric field $\mathbf{E} = E\mathbf{e}_x$ and a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$, where E and B are constants. Find the equations of motion for the electron.

Book 2 Chapter 2

Exercise 2.1 A sphere of radius $1.50\,\mathrm{mm}$ consists of liquid argon, atomic number $Z = 18$ and number of atoms per unit volume $n = 2.17 \times 10^{28}\,\mathrm{m^{-3}}$. When placed in a uniform electric field, it is found that the total dipole moment $\mathbf{\Pi}$ of the sphere has a magnitude of $5.50 \times 10^{-14}\,\mathrm{C\,m}$.

(a) Calculate the average dipole moment of an argon atom in this particular electric field, and the corresponding average separation of the centre of charge of the electron cloud and the argon nucleus.

(b) Find the polarization $\mathbf{P}$ of the sphere.

Exercise 2.2 A dielectric sphere of radius R has polarization $\mathbf{P} = ar\cos^2\theta\,\mathbf{e}_r - ar\cos\theta\,\sin\theta\,\mathbf{e}_\theta$, where a is a constant and r, θ, ϕ are spherical coordinates whose origin is the centre of the sphere.

(a) Find the bound volume charge density ρ_b within the sphere.

(b) What is the bound surface charge density σ_b on the surface of the sphere?

Exercise 2.3 A slab of dielectric material, with uniform relative permittivity $\varepsilon = 6.0$, is placed in air perpendicular to a uniform electric field $\mathbf{E}$ of magnitude $12\,000\,\mathrm{V\,m^{-1}}$.

Inside the dielectric, what are the magnitudes of

(a) the electric field,

(b) the electric displacement,

(c) the polarization?

You should assume that $\varepsilon_\mathrm{air} = 1$.

Exercise 2.4 A parallel plate capacitor has plates with area A and separation $d_1 + d_2$. Two slabs of dielectric materials of relative permittivities ε_1 and ε_2 and thicknesses d_1 and d_2 fill the gap between the plates. The free charge per unit area on plate 1 is σ. There is no accumulation of free charge at the interface between the dielectrics.

(a) Neglecting any edge effects, what is the free charge per unit area on the lower plate? Calculate the magnitudes of the electric displacement $\mathbf{D}$, the electric field $\mathbf{E}$ and the polarization $\mathbf{P}$ in the dielectric materials between the plates.

(b) Find expressions for the potential difference across each dielectric and the capacitance of the capacitor.

Book 2 Chapter 3

Exercise 3.1 A long cylindrical wire of radius R carries a steady current with uniform current density of magnitude J. The wire is made of a material with relative permeability μ. Find the magnetic intensity $\mathbf{H}$ and the magnetic field $\mathbf{B}$ at a point *inside* the wire a distance r from the wire's central axis.

Exercise 3.2 State the boundary conditions for the $\mathbf{B}$ and $\mathbf{H}$ fields at the plane boundary between two media in the absence of free currents at the interface.

A uniform magnetic field $\mathbf{B}$ crosses a plane boundary between two LIH non-conducting media. In the first medium, of relative permeability $\mu_1 = 1.5$, the $\mathbf{B}$ field is at an angle of $60°$ to the normal to the boundary. In the second medium, of relative permeability $\mu_2 = 3.0$, the $\mathbf{B}$ field is at an angle of θ to the normal to the boundary. Calculate the value of θ.

Exercise 3.3 A coil has 2000 turns wound on a toroidal ring of soft iron alloy of relative permeability 400. The ring has a mean radius of 12 cm (measured to the centre of the iron).

(a) Assuming that the flux is restricted to the iron and that $\mathbf{B}$ has constant magnitude within the iron, calculate the magnitude of $\mathbf{H}$ and $\mathbf{B}$ within the iron when a current of 1.5 A flows in the coil.

(b) Suppose now that a break is made in the iron so that there is a uniform air-gap of width 3.0 mm. Calculate the magnitudes of $\mathbf{H}$ and $\mathbf{B}$ in this case, both in the iron and in the gap.

Exercise 3.4 A sample of a linear, isotropic, homogeneous material is placed in a static magnetic field $\mathbf{B}(\mathbf{r})$. Show that the bound current density $\mathbf{J}_\mathrm{b}(\mathbf{r})$ is proportional to the free current density $\mathbf{J}_\mathrm{f}(\mathbf{r})$. Hence show that when no free currents flow through the sample, the bound current density vanishes, ensuring that all magnetization currents are confined to the surface.

Book 2 Chapter 4

Exercise 4.1 Identify each of the following equations, and state the circumstances in which you would apply each of them to solve an electrostatic field problem.

(a) $\operatorname{div}\mathbf{E} = \dfrac{\rho}{\varepsilon_0}$

(b) $\operatorname{div}\mathbf{D} = \rho_\mathrm{f}$

(c) $\int_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \dfrac{1}{\varepsilon_0}\int_\tau \rho\,\mathrm{d}\tau$

(d) $\int_S \mathbf{D}\cdot \mathrm{d}\mathbf{S} = \int_\tau \rho_\mathrm{f}\,\mathrm{d}\tau$

(e) $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$

(f) $\nabla^2 V = -\dfrac{\rho_\mathrm{f}}{\varepsilon\varepsilon_0}$

(g) $\nabla^2 V = 0$

Exercise 4.2 Consider a pair of parallel conducting plates, 1 and 2, with plate 1 in the plane $x = 0$ and held at a fixed potential of zero volts, and plate 2 in the plane $x = s$ at a fixed potential V_s. Between the plates is a dielectric material with relative permittivity ε in which there is fixed a fixed charge density $\rho_\mathrm{f}(x) = A + Bx$, where A and B are constants. The plates are so large compared to their separation s that the electric field between them can be considered to depend only on x, the perpendicular distance from plate 1 towards plate 2.

Show that the potential in the dielectric material is

$$V(x) = V_s\,\frac{x}{s} - \frac{1}{\varepsilon\varepsilon_0}\left[\tfrac{1}{2}A(x^2 - sx) + \tfrac{1}{6}B(x^3 - s^2x)\right],$$

and determine the electrostatic field $\mathbf{E}$ in the material.

Exercise 4.3 A solid metal sphere of radius a is concentric with a hollow metal sphere of inner radius $2a$. The space between the spheres is filled with a dielectric material of relative permittivity ε carrying a uniform static charge density ρ_0. The charge density is zero outside the spheres.

(a) Write down Poisson's equation in spherical coordinates for the electrostatic potential V in the region $a < r < 2a$ between the spheres.

(b) Find the general solution of Poisson's equation in the region between the spheres.

(c) Now assume that the two metal spheres are held at zero potential. Use the boundary conditions, together with your answer to part (b), to show that

$$V(r) = -\frac{\rho_0 r^2}{6\varepsilon\varepsilon_0}\left[1 + 6\left(\frac{a}{r}\right)^3 - 7\left(\frac{a}{r}\right)^2\right]$$

$$(a \le r \le 2a)\,.$$

Exercise 4.4 Two equal point charges q are located at points with Cartesian coordinates $(d/2,\, d/2,\, 0)$ and $(d/2,\, -d/2,\, 0)$ in front of an infinite conducting plate at zero potential in the plane $x = 0$. Use the method of images to find the electric force acting on the charge at $(d/2,\, d/2,\, 0)$, and hence show that the magnitude of this force is $3q^2/8\pi\varepsilon_0 d^2$.

Exercise 4.5 (a) Show that the potential

$$V(r,\,\phi,\,z) = Ar\cos\phi + \frac{B\cos\phi}{r}$$

is a solution of Laplace's equation in cylindrical coordinates.

(b) An infinitely long dielectric cylinder, with radius R and relative permittivity ε, is placed with its axis perpendicular to a uniform electric field $\mathbf{E}_0$. Find the electrostatic potential inside and outside the cylinder.

Exercise 4.6 Obtain an equation, like Equation 4.26 on p. 88 of Book 2, for Laplace's equation in *three* dimensions.

Exercise 4.7 Consider the numerical solution of Laplace's equation for a square region with $V = 0$ on the top and bottom sides and $V = 1$ (volt) on the left and right sides. Use the Jacobi relaxation numerical method to determine the values of the potential on a nine point square grid within the square region. The spacing of the grid points is $\frac{1}{4}$ of the side length of the square region, and the grid is centrally located within the region (see Figure 5). Use the starting values for the potential at the internal grid points as shown in the figure, and work to three significant figures. Note that the bold values, **0** and **1**, are the boundary potentials, and are therefore fixed.

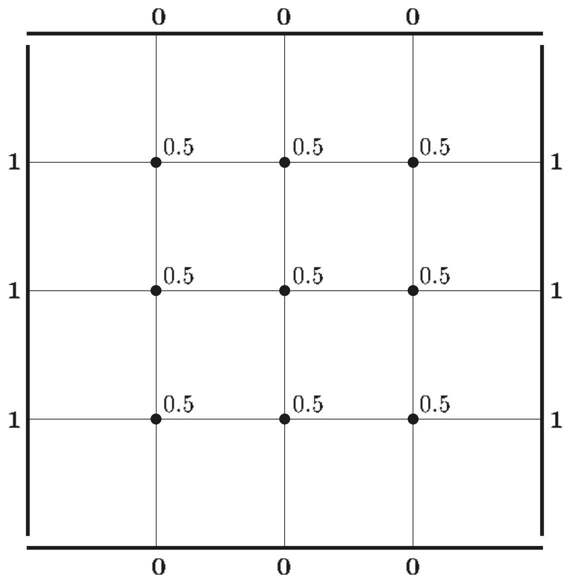

Figure 5 For Exercise 4.7.

Book 2 Chapter 5

Exercise 5.1 An infinitesimal current element $I\,\delta\mathbf{l}'$ is located at the point $(0.02,\ 0,\ 0)$ m in a Cartesian coordinate system. It points in the direction $\mathbf{e}_y$ and has magnitude 5.0×10^{-6} A m. What is the magnetic field due to this current element at the point $(0.08,\ 0.08,\ 0)$ m?

Exercise 5.2 Derive an expression for the magnitude of the magnetic field at the centre of a regular hexagonal loop with sides of length L, carrying current I.

Exercise 5.3 (a) Show that the axial field at one end of a long solenoid is approximately half that in the middle.

(b) Show that the axial field at the middle of a solenoid that is ten times longer than its diameter differs from the value for a solenoid of infinite length by less than 1%.

Exercise 5.4 (a) Find the magnetic field corresponding to the magnetic vector potential

$$\mathbf{A} = -\tfrac{1}{2}\,a\,(y^2\mathbf{e}_x + z^2\mathbf{e}_y + x^2\mathbf{e}_z).$$

(b) What is the current density associated with this magnetic field?

Exercise 5.5 A magnetometer is used to measure the radial component of the magnetic field outside the (perfectly spherical) head of a patient. A current dipole is located a distance d directly below the highest point of the sphere (head) and the radial field is measured just outside the scalp, directly above the highest point. What is the measured field due to the combination of dipole and return currents in the following three cases?

(a) The dipole is perpendicular to the surface of the sphere at its closest point.

(b) The dipole is parallel to the surface of the sphere at its closest point.

(c) The dipole is at $45°$ to the surface of the sphere at its closest point.

Exercise 5.6 For the situation described in Exercise 5.5b, describe how the radial magnetic field varies over the scalp in the region around the highest point of the head, and indicate what information can be deduced from the measured radial field component.

Book 2 Chapter 6

Exercise 6.1 An electron is travelling at velocity $\mathbf{v} = 2.0\times10^6\,\mathrm{m\,s^{-1}}(\mathbf{e}_y+\mathbf{e}_z)$ when it passes through a point where the electric field is $\mathbf{E} = 2.5\times10^6\,\mathrm{V\,m^{-1}}(\mathbf{e}_x+2\mathbf{e}_y)$ and the magnetic field is $\mathbf{B} = 2.5\,\mathrm{T}\,(\mathbf{e}_x+\mathbf{e}_z)$. What is the magnitude of the resultant force on the particle at this point?

Exercise 6.2 A proton is released from rest at time $t=0$ in a region where both the electric field and the magnetic field are uniform and steady. Use the following information to determine the magnitudes and directions of the fields.

(i) The initial acceleration of the proton is in the $+x$-direction, and has magnitude $a_0 = 9.6\times10^{14}\,\mathrm{m\,s^{-2}}$.

(ii) At a subsequent time t_1, the proton is travelling at speed $v_1 = 2.0\times10^7\,\mathrm{m\,s^{-1}}$ in the $-y$-direction and has an acceleration of magnitude a_0 in the $-x$-direction.

(iii) The proton's motion takes place in a plane with constant z.

Exercise 6.3 Velocity selectors are used in particle accelerators to remove particles from a high energy beam that do not have a specified velocity. These devices use a combination of electric and magnetic fields, which you may assume are uniform in the region of interest, to deflect particles that are not travelling at the required velocity, and to allow particles with the required velocity to pass undeflected through an exit aperture. If the beam is travelling in the $\mathbf{e}_y$-direction and the magnetic field is $0.50\,\mathrm{T}$ in the $\mathbf{e}_z$-direction, what electric field is required to select particles travelling at 10% of the speed of light?

Exercise 6.4 A magnetic field of magnitude roughly $B_{\mathrm{gal}} = 2\times10^{-10}\,\mathrm{T}$ is believed to exist throughout our galaxy. Suppose that a cosmic ray proton travelling at about 10% of the speed of light is trapped in this uniform field. What are the period and the diameter of its orbit? (Assume that a particle travelling at 10% of the speed of light can be treated using non-relativistic mechanics.)

Exercise 6.5 Mass spectrometers separate ions according to their velocity and charge-to-mass ratio. In a particular design of mass spectrometer, ions

travelling at velocity $\mathbf{v} = v\,\mathbf{e}_z$ in a field-free region are injected at the origin of a Cartesian coordinate system into a region in which uniform electric and magnetic fields act in the $\mathbf{e}_y$-direction, that is, $\mathbf{E} = E\mathbf{e}_y$ and $\mathbf{B} = B\mathbf{e}_y$. The ions are deflected by the fields and fall onto a detector grid in the plane $z = 0$.

(a) If the mass of an ion is m, and its charge Q, show that its equation of motion can be written as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{QB}{m}\frac{\mathrm{d}z}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{QE}{m}, \quad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \frac{QB}{m}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

(b) Hence show that

$$x = \frac{mv}{QB}\left[\cos\left(\frac{QB}{m}t\right) - 1\right], y = \frac{QE}{2m}t^2,$$
$$z = \frac{mv}{QB}\sin\left(\frac{QB}{m}t\right).$$

(c) Show that an ion of mass m, charge Q and initial velocity $\mathbf{v}$ is collected at the point on the detector grid (in the $z = 0$ plane) with coordinates

$$\left[-\frac{2mv}{QB},\ \frac{\pi^2 mE}{2QB^2},\ 0\right].$$

Exercise 6.6 An electron spirals around the magnetic field lines produced by a long straight wire carrying 100 A in a large vacuum chamber. The wire is located on the z-axis of cylindrical coordinates, and the electron has a velocity component in the direction of the magnetic field of $v_\phi \equiv v_\| = 1.0 \times 10^5\,\mathrm{m\,s^{-1}}$ and a velocity component perpendicular to the field direction of $v_\perp = 1.0 \times 10^5\,\mathrm{m\,s^{-1}}$. Starting from Equations 6.14 and 6.20 of Book 2, derive expressions for the drift velocity of the electron (i) due to the curvature of the magnetic field lines, and (ii) due to the variation of field strength with position, in terms of I, $v_\perp$, $v_\|$ and fundamental constants. Hence calculate the net drift speed for the given data.

Exercise 6.7 A magnetic bottle is axially symmetric about the z-axis of a cylindrical coordinate system. Near the middle of the bottle the magnetic field is uniform, has a magnitude of 2.0×10^{-3} T and is in the $+z$-direction. At $t = 0$, an electron at a point on the plane $z = 0$ has velocity $\mathbf{v} = v_\phi\,\mathbf{e}_\phi + v_z\,\mathbf{e}_z$, where $v_\phi = 3.0 \times 10^6\,\mathrm{m\,s^{-1}}$ and $v_z = 2.0 \times 10^6\,\mathrm{m\,s^{-1}}$.

(a) What is the initial radius of the electron's orbit?

(b) What is the magnitude of the magnetic field at the mirror point for this electron?

(c) What is the radius of the electron's orbit at the mirror point?

(d) What is the velocity of the electron when it returns to the plane $z = 0$?

Book 2 Chapter 7

Exercise 7.1 Two parallel-sided slabs, 1 and 2, of materials with low conductivities, σ_1 and σ_2, have thicknesses, d_1 and d_2, and the same area. One face of each slab is in good electrical contact with a face of the other slab, and the two external faces of the combined slab are coated with thick layers of a high-conductivity metal.

If the potential difference between the metal layers is ΔV, with the metal layer on slab 1 at Earth potential, $V = 0$, neglecting any edge effects, what is the potential at the interface between the two materials? Check that the expression you derive gives reasonable results for limiting values of the conductivities and thicknesses of the slabs.

Exercise 7.2 Electric currents are set up in a steel tube in three different ways.
(a) The ends of the tube are clamped between two copper blocks, and the current flows along the length of the tube.
(b) The tube is filled with a high-conductivity fluid, its ends are capped with insulating material, and the tube immersed in a high-conductivity fluid. A current is passed between the two fluids through the walls of the tube.
(c) A magnetic field that increases linearly with time is applied parallel to the axis of the tube so that a current is induced to flow around the wall of the tube.

Calculate the resistance to current flow in these three situations for a tube that is 10 cm long, has radius 5.0 mm and wall thickness 0.50 mm, and conductivity $1.0 \times 10^7\,\Omega^{-1}\,\mathrm{m}^{-1}$.

Exercise 7.3 Solve the problem illustrated in Figure 7.9 of Book 2, by deriving an expression for the resistance between a small sphere, radius a, embedded in a large block of poorly conducting material, and a metal coating on one face of the block. The sphere is at a distance d from the metal surface. Hence calculate the resistance for a semiconducting block, $\sigma = 5.0\,\Omega^{-1}\,\mathrm{m}^{-1}$, if the ratio $d/a = 10$.

Exercise 7.4 Consider a situation similar to the 'jumping ring' demonstration described in Exercise 7.6 and Figure 7.15 on p. 157 of Book 2, but with no iron core in the solenoid. What forces act on the aluminium ring when the current is switched on (a) when the ring is horizontal and just above the centre of the solenoid, and (b) when the ring is horizontal and just above the top end of the solenoid.

Exercise 7.5 A circular coil, with N turns, area A, rotates at frequency f about a vertical axis that lies along a diameter of the coil. (See Figure 6). The amplitude of the voltage across the terminals of the coil is V_0. Derive an expression for the magnitude B_{horiz} of the field in the horizontal plane.

Hence determine the value of B_{horiz} if $N = 50$, $A = 0.50 \times 10^{-4}\,\text{m}^2$, $f = 80\,\text{Hz}$ and $V_0 = 25\,\text{mV}$.

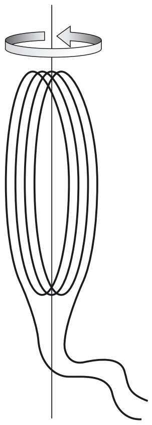

Figure 6 For Exercise 7.5.

Exercise 7.6 (a) Two coils, A and B, are in close proximity to each other. When a current $I_\text{A} = I_{\text{A}0} \sin \omega t$ flows through coil A, an induced current $I_\text{B} = -I_{\text{B}0} \cos \omega t$ flows through a large resistance R_B connected across the terminals of coil B. Derive an expression for the mutual inductance M between the two coils in terms of parameters introduced above. (You should assume that $R_\text{B} \gg \omega L_\text{B}$, where L_B is the self-inductance of coil B.)

(b) The current source is disconnected from coil A and a large resistance R_A is connected between the terminals of the coil. If a current $I_B = I'_{\text{B}0} \exp\left[-\alpha t\right]$ flows through coil B, what is the current in coil A? (Assume that $R_\text{A} \gg \omega L_\text{A}$.)

Exercise 7.7 An inductor core is made by drilling a hole, radius a, through the centre of a disc that has thickness a, radius $2a$ and relative permeability μ. A toroidal coil with N turns of wire is tightly wound on the core. Derive an expression for the self-inductance L of the inductor, and hence calculate the value of L if the inductor has 100 turns, with $a = 4.0\,\text{mm}$ and $\mu = 500$. (Assume the core is an LIH material.)

Book 2 Chapter 8

Exercise 8.1 Four equal charges, q, are located at the points $(a,\,0,\,0)$, $(0,\,a,\,0)$, $(-a,\,0,\,0)$, $(0,\,-a,\,0)$. (see Figure 7). What is the electrostatic energy of this arrangement?

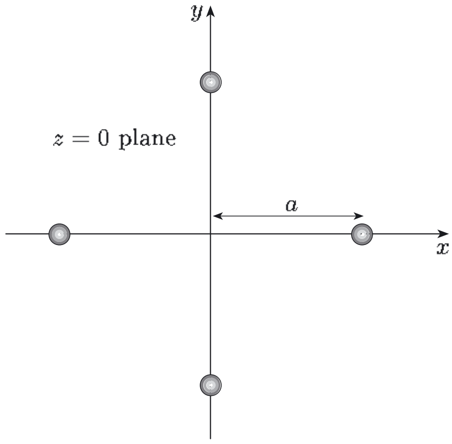

Figure 7 For Exercise 8.1.

Exercise 8.2 (a) What capacitance C is required to store $5.0\,\text{MJ}$, if the maximum voltage difference across the capacitor is $2.0\,\text{kV}$?

(b) What inductance L is required to store $5.0\,\text{MJ}$, if the maximum current through the inductor is $2.0\,\text{kA}$?

(c) What are the time constants for charging the capacitor through a resistance R of $0.2\,\Omega$ and for establishing the current in the inductor through the same resistance?

(d) What is the resonant frequency f_n of a circuit in which components with these values of C, L and R are connected in series?

Exercise 8.3 What is the ratio of the electrical energy to the magnetic energy stored in unit volume of the atmosphere just above the Earth's surface, where the electric field strength is $100\,\text{V}\,\text{m}^{-1}$ and the magnetic field strength is $5.0 \times 10^{-5}\,\text{T}$?

Exercise 8.4 An electronic flash unit contains a $1000\,\mu\text{F}$ capacitor and is charged using a $300\,\text{V}$ power supply.

(a) It takes $5\,\text{s}$ to charge the capacitor to 95% of the maximum charge. What resistance R is connected in series with the capacitor in the charging circuit?

(b) 90% of the energy stored in the capacitor is discharged in $2\,\text{ms}$. What is the average power dissipated during this period?

Exercise 8.5 Show that the energy U stored in the electric field in the free space surrounding a conducting sphere, radius R, carrying charge Q, is given by

$$U = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

How does this expression compare with the electric potential energy of the charge on a conducting sphere, determined in Exercise 8.2 in Book 2?

Exercise 8.6 In Worked Example 8.1 of Book 2, we showed that the electric potential energy of a uniformly charged sphere, with radius R and charge

density ρ, is given by

$$U = \frac{4\pi\rho^2 R^5}{15\varepsilon_0}.$$

(a) Obtain an expression for U in terms of the total charge Q in the sphere.

(b) Write down an expression for the energy stored in the electric field in the region outside the sphere. (*Hint*: you can make use of a result from the previous question.)

(c) Explain why the expressions in parts (a) and (b) are different, and confirm your explanation by independently deriving an expression for the difference in energy.

Exercise 8.7 A naive inventor plans to store energy by passing a current through a large air-filled solenoid, $1.0\,\mathrm{m}$ long, $10\,\mathrm{cm}$ radius. The solenoid will be wound with 200 turns of copper wire, radius $2.5\,\mathrm{mm}$, conductivity $6.5\times10^7\,\Omega^{-1}\,\mathrm{m}^{-1}$. Calculate the energy that would be stored by this solenoid when carrying a current of $100\,\mathrm{A}$ and the power dissipated, and hence comment on the feasibility of this proposal.

Book 2 Chapter 9

Exercise 9.1 A circular coil, with N turns of radius a and with self-inductance L, is wound from superconducting wire. The start of the first turn is connected to the end of the Nth turn to form a continuous superconducting path. The coil is cooled below its critical temperature in a uniform applied magnetic field $\mathbf{B}_0$, which is perpendicular to the plane of the coil, and the applied magnetic field is then reduced to zero.

(a) What is the magnetic flux through the coil after the applied field is reduced to zero?

(b) What is the current in the coil?

(c) What is the field at the centre of the coil?

(d) What can you say about the magnetic field in the plane of the coil at distance $0.95R$ from its centre?

Exercise 9.2 (a) Estimate the temperature below which a long thin cylinder of indium is superconducting in a magnetic field with strength $B = 0.014\,\mathrm{T}$, aligned parallel to its axis. (You will need to use data from Chapter 9 of Book 2.)

(b) Estimate the critical current of a long straight indium wire, radius $1.0\,\mathrm{mm}$, at $T = 1.2\,\mathrm{K}$ in the Earth's magnetic field.

Exercise 9.3 Two superconducting discs with identical dimensions are made from two different superconductors. Disc A is made from a material that has a penetration depth $\lambda_\mathrm{A} = 40\,\mathrm{nm}$ and a coherence length of $\xi_\mathrm{A} = 800\,\mathrm{nm}$, whereas disc B has a penetration depth of $\lambda_\mathrm{B} = 80\,\mathrm{nm}$ and a coherence length $\xi_\mathrm{B} = 4\,\mathrm{nm}$. Each disc is placed in turn in a solenoid, with the plane of the disc perpendicular to the solenoid's axis, and the magnitude of the field in the solenoid is steadily increased. Describe what happens to the magnetic field in and around the discs as the magnitude of the field in the solenoid is increased.

Exercise 9.4 Produce a table that briefly contrasts type-I and type-II superconductors.

Exercise 9.5 A long thin superconducting cylinder with radius R is aligned with its axis parallel to the direction of a uniform applied magnetic field $\mathbf{B}_0 = B_{0z}\mathbf{e}_z$.

(a) Using Equation 9.12 of Book 2 as a guide to how the magnetic field varies near to the surface of a superconductor, write down an expression in cylindrical coordinates for the field within the superconducting cylinder. Note any assumptions that you make.

(b) Use Ampère's law to derive an expression for the current density within the cylinder.

(c) Derive an expression for the screening current per unit length flowing near the surface of the cylinder, and hence confirm that this current produces a uniform magnetic field within the long cylinder that cancels the applied field.

Exercise 9.6 The mean magnetic field in a certain type-II superconductor in the mixed state is $0.50\,\mathrm{T}$. Given that the flux within each normal core is equal to the flux quantum ($2.07\times10^{-15}\,\mathrm{T\,m^2}$), what is the number density of cores passing through a plane perpendicular to the field direction? What is the average separation between neighbouring cores?

Book 2 Chapter 10

Exercise 10.1 Show that $E^2 - c^2B^2$ is an invariant quantity.

Exercise 10.2 In the laboratory frame of reference, $\mathcal{F}$, there is an electric field of magnitude E in the $+x$-direction and a magnetic field of magnitude B inclined at an angle θ to the $+x$-direction.

(a) What are the electric and magnetic fields in a frame $\mathcal{F}'$ that is travelling at uniform speed v in the x-direction relative to the laboratory frame? Determine the inclination of the magnetic field to the $+x$-direction in the frame $\mathcal{F}'$. (To simplify the derivation, assume that the magnetic field has no component in the z-direction.)

(b) If $\theta = 30^\circ$, at what speed v must frame $\mathcal{F}'$ travel relative to frame $\mathcal{F}$ for the magnetic field observed in this frame to be at 60° to the x'-axis?

Exercise 10.3 Which of the following are invariant quantities?

- the rest mass of an electron;
- the charge of an electron;
- the speed of an electron;
- the energy of an electron;
- the charge density of a cloud of electrons;
- the speed of light in vacuum;
- $\mathbf{E} \cdot \mathbf{B}$.

Exercise 10.4 Throughout a certain region of space there is a stationary distribution of positive charge, with uniform density ρ_+, and a distribution of negative charge, with uniform density ρ_-, that is travelling with speed v in the $+x$-direction. The net charge density is zero, that is $\rho_+ = -\rho_-$.

(a) What are the current densities, $\mathbf{J}_+$ and $\mathbf{J}_-$, due to the positive charge distribution and due to the negative charge distribution?

(b) What are the charge densities and current densities due to the positive charge distribution and due to the negative charge distribution measured by an observer who is travelling at speed v in the $+x$-direction? What are the net charge density and current density according to this observer?

Exercise 10.5 An infinite cylinder with cross-sectional area A and uniform charge density ρ is at rest along the x-axis of an inertial frame $\mathcal{F}$. A test charge Q is stationary at point $(0,\, d,\, 0)$ outside the cylinder.

(a) What are the electric and magnetic forces acting on charge Q according to an observer in frame $\mathcal{F}$?

(b) A second frame, $\mathcal{F}'$, is in standard configuration with frame $\mathcal{F}$, and travels at speed v along its x-axis. What are the charge density and current density associated with the cylinder according to an observer in frame $\mathcal{F}'$?

(c) What are the electric and magnetic fields generated by the cylinder according to the observer in frame $\mathcal{F}'$?

(d) What are the electric and magnetic forces on test charge Q according to an observer in frame $\mathcal{F}'$, and what is the total force on Q?

(e) Check that the net force in frame $\mathcal{F}'$ determined in (d) is the same as that predicted by the force transformation relationships in Equations 10.17–10.19 of Book 2.

Exercises for Book 3

Book 3 Chapter 1

Exercise 1.1 (a) Write down an expression for the physical electric field of a plane wave that is travelling in the direction $(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)/\sqrt{3}$ with amplitude E_0, wavenumber k, frequency f, phase shift 0 and polarization in the direction $(\mathbf{e}_x - \mathbf{e}_y)/\sqrt{2}$.

(b) Confirm that your expression represents a transverse wave.

(c) Write down another possible linear polarization for a transverse wave travelling in the direction specified in part (a).

Exercise 1.2 The expression

$$\mathbf{E} = \alpha \cos\left(3\beta x + 2\beta y + \beta z + \gamma t\right)(\mathbf{e}_x - 3\mathbf{e}_y)$$

represents the electric field of a plane monochromatic electromagnetic wave. If the values of the constants in this expression are $\alpha = 2.0 \times 10^2\,\mathrm{V\,m^{-1}}$, $\beta = 6.0 \times 10^6\,\mathrm{m^{-1}}$ and $\gamma = 6.7 \times 10^{15}\,\mathrm{s^{-1}}$, what are the wavenumber, frequency and amplitude of this wave? Confirm that the wave travels at speed c, the phase speed for electromagnetic waves in a vacuum.

Exercise 1.3 An electromagnetic wave is travelling in the direction $\frac{1}{3}(2\mathbf{e}_x + 2\mathbf{e}_y - \mathbf{e}_z)$ and is polarized in the direction $(\mathbf{e}_y + 2\mathbf{e}_z)/\sqrt{5}$. In what direction is the magnetic field associated with this wave?

Exercise 1.4 What types of waves are produced by combining the plane wave

$$\mathbf{E} = E_0 \exp[\mathrm{i}(kx - \omega t)]\,\mathbf{e}_y$$

in turn with each of the following plane waves?

(a) $E_0 \exp[\mathrm{i}(kx - \omega t)]\,\mathbf{e}_y$

(b) $E_0 \exp[\mathrm{i}(kx - \omega t)]\,\mathbf{e}_z$

(c) $2E_0 \exp[\mathrm{i}(kx - \omega t)]\,\mathbf{e}_z$

(d) $E_0 \exp[\mathrm{i}(kx - \omega t + \pi/2)]\,\mathbf{e}_z$

(e) $E_0 \exp[\mathrm{i}(kx - \omega t - \pi/2)]\,\mathbf{e}_z$

(f) $2E_0 \exp[\mathrm{i}(kx - \omega t + \pi/2)]\,\mathbf{e}_z$

(g) $E_0 \exp[\mathrm{i}(kx - \omega t - \pi)]\,\mathbf{e}_y$

(h) $E_0 \exp[\mathrm{i}(kx + \omega t)]\,\mathbf{e}_y$

(i) $E_0 \exp[\mathrm{i}(1.01kx - 1.01\omega t)]\,\mathbf{e}_y$

Exercise 1.5 A powerful research laser produces pulses of light, wavelength $527\,\mathrm{nm}$, containing $300\,\mathrm{J}$ of energy in $1.0 \times 10^{-9}\,\mathrm{s}$, and concentrated into an area with diameter $1.0 \times 10^{-4}\,\mathrm{m}$. Assume that the pulse is uniform over its area and uniform in time over its nanosecond duration, and that it is polarized in the x-direction and is travelling in the $+y$-direction through a vacuum.

(a) Calculate all of the following quantities, which specify the radiation during the pulse: ω, k_x, k_y, k_z, E_{0x}, E_{0y}, E_{0z}, B_{0x}, B_{0y} and B_{0z}.

(b) Given that air is ionized by an electric field strength exceeding $3 \times 10^6\,\mathrm{V\,m^{-1}}$, explain why the beam must travel through a vacuum.

Exercise 1.6 A linearly polarized plane wave travelling in the horizontal direction has its electric field, amplitude E_0, in the vertical direction. The wave passes through a polaroid filter that has its axis

of polarization at 30° to the vertical, and is then incident on a second polaroid filter. What is the amplitude of the electric field transmitted by the second filter when the axis of polarization of this filter is (a) vertical, and (b) horizontal? Assume that the filters are 'ideal', that is, they are perfectly transparent to waves polarized parallel to the axis of polarization and perfectly opaque to waves polarized perpendicular to the axis of polarization.

Exercise 1.7 (a) In Chapter 1 of Book 3 we showed that the time-averaged value of the magnitude of the Poynting vector for a monochromatic plane wave is $\overline{N} = \frac{1}{2}\varepsilon_0 E_0^2 c$. Show that an alternative expression is $\overline{N} = \frac{1}{2}B_0^2 c/\mu_0$.

(b) If the power per unit area of a radio wave in the region of a radio receiver is $10^{-10}\,\mathrm{W\,m^{-2}}$, what is the amplitude of the magnetic field associated with the wave?

Book 3 Chapter 2

Exercise 2.1 For each of the following oscillating current elements, state whether it can be regarded as a Hertzian dipole, and for those that can, indicate how far away from the dipole the point of measurement must be if the fields are to be well represented by the radiation field alone.

(a) Dipole length $10\,\mathrm{km}$, oscillating at $50\,\mathrm{Hz}$.

(b) Dipole length $10\,\mathrm{m}$, oscillating at $100\,\mathrm{MHz}$.

(c) Dipole length $1\,\mathrm{cm}$, oscillating at $3\,\mathrm{GHz}$.

Exercise 2.2 Show that Equation 2.17 of Book 3 reduces to Equation 2.1 in the limit when $\omega \to 0$ and r is small.

Exercise 2.3 The vector amplitude of the electric field at a point with spherical coordinates $(1.0\,\mathrm{m},\ \pi/2,\ 0)$ at time $t = 0$ due to a Hertzian dipole, frequency $3\,\mathrm{GHz}$, located at the origin and aligned with the $\theta = 0$ direction, is $2.0 \times 10^{-6}\,\mathrm{V\,m^{-1}}\,\mathbf{e}_\theta$.

(a) What is the vector amplitude of the electric field at $(2.0\,\mathrm{m},\ \pi/4,\ \pi/2)$?

(b) What is the vector amplitude of the magnetic field at $(2.0\,\mathrm{m},\ \pi/4,\ \pi/2)$?

Exercise 2.4 A Hertzian dipole is located at the origin of Cartesian coordinates and is aligned with the y-axis. The strength of the dipole is $3.0 \times 10^{-8}\,\mathrm{A\,m}$, and its angular frequency is $2.0 \times 10^9\,\mathrm{s^{-1}}$. Calculate the time-averaged value of the Poynting vector at the point with coordinates $(2.0,\ -2.0,\ 1.0)\,\mathrm{m}$.

Exercise 2.5 The mean scattering cross-section for blue light with $\lambda = 400\,\mathrm{nm}$ by air molecules is $1.50 \times 10^{-30}\,\mathrm{m^2}$. Determine the percentage of the incident power that is scattered when blue light passes through $1.0\,\mathrm{km}$ of air at sea-level $(P = 1.0 \times 10^5\,\mathrm{Pa})$ at $0\,^\circ\mathrm{C}$, and compare this with the percentage scattered for red light, $\lambda = 600\,\mathrm{nm}$, under the same conditions. You should assume that there is negligible scattering by water droplets and dust particles.

Exercise 2.6 Assume that you are lying on your back on a beach under a cloudless sky with the Sun directly overhead and your feet pointing towards the south. A tiny sunshade blocks direct light from the Sun reaching your eyes, but otherwise all of the sky is visible. You use a polarizing filter that completely blocks the component of the incident light with polarization parallel to one edge of the filter but transmits the component with polarization perpendicular to this edge. Explain what you would expect to observe when looking at the sky through this filter.

Exercise 2.7 A Hertzian dipole is located at the origin of spherical coordinates and is aligned with the direction $\theta = 0$.

(a) What is the ratio of the time-average power flow per unit area at a point with coordinates $(50\,\mathrm{m},\ 90°,\ 0)$ to that at $(100\,\mathrm{m},\ 60°,\ 90°)$?

(b) What fraction of the total power is radiated in the range $70° < \theta < 110°$?

Book 3 Chapter 3

Exercise 3.1 Electromagnetic radiation with angular frequency $\omega = 2\pi \times 10^8\,\mathrm{s^{-1}}$ travels through a material with relative permittivity $\varepsilon = 4.0$. What are the values of

(a) the frequency,

(b) the phase speed,

(c) the wavelength, and

(d) the wavenumber for the radiation in this material?

Exercise 3.2 An electromagnetic plane wave travelling in free space, with electric field given by

$$\mathbf{E}_\mathrm{i} = E_0 \exp[\mathrm{i}(ky - \omega t)]\,\mathbf{e}_z,$$

is normally incident on a boundary, in the plane $y = 0$, of a dielectric material with relative permittivity ε.

(a) Write down expressions for the magnetic field associated with this electric field and for the electric and magnetic fields of the waves transmitted and reflected at the boundary. You should express these fields in terms of E_0, k, ω, ε and the ratios r_n and t_n of the amplitudes of the reflected and transmitted electric fields to the amplitude of the incident electric field.

(b) By applying the boundary conditions for the electric and magnetic fields at the interface in the plane $y = 0$, derive expressions for the amplitude reflection ratio and the amplitude transmission ratio for normally incident radiation.

Exercise 3.3 A plane electromagnetic wave travelling through free space is normally incident on

the surface of a dielectric material with relative permittivity 1.7.

(a) What is the value of the refractive index of the material, and what is the value of the reflectance of the surface for this wave?

(b) If the amplitude of the electric field of the incident wave is $5.0\,\mathrm{V\,m^{-1}}$, determine the amplitudes of the electric fields of the reflected and transmitted waves, and show that your results satisfy the boundary condition for **E**.

Exercise 3.4 What percentage of the power of normally incident light is transmitted through the glass wall of a fish tank filled with water (a) when the light travels from the surrounding air into the water, and (b) when the light travels from the water into the air? Assume $n_{\text{glass}} = 1.50$ and $n_{\text{water}} = 1.33$.

Exercise 3.5 Linearly polarized light travelling through air is incident at $45°$ to the normal to a plane face of a diamond, which has refractive index $n = 2.42$. The polarization direction is at $45°$ to the scattering plane. What is the total reflectance in this situation?

Exercise 3.6 Circularly polarized light is incident at the Brewster angle on a dielectric surface. Describe the polarization of the light that is reflected from the surface.

Exercise 3.7 A thin parallel-sided glass plate with refractive index 1.55 is sandwiched between large blocks of glass with refractive index 1.50. An infrared beam travels in the thin plate at an angle θ_i to the normal to its surface.

(a) For what range of values of θ_i will the wave be essentially confined within the plate?

(b) For an infrared beam with a free space wavelength of $1500\,\text{nm}$ that travels at $80°$ to the normal, what is the characteristic distance that the electric field penetrates into the blocks on either side of the plate?

Book 3 Chapter 4

Exercise 4.1 A plane wave travels in the $\mathbf{e}_y$-direction through a dielectric medium with refractive index $n = n_{\text{real}} + \mathrm{i}\, n_{\text{imag}}$. What is the ratio of the amplitudes of the magnetic fields at the points with Cartesian coordinates $(x_1,\, y_1,\, z_1)$ and $(x_2,\, y_2,\, z_2)$?

Exercise 4.2 The physical electric field associated with a plane wave in a dielectric material has the form

$$\mathbf{E}_{\text{phys}} = E_0 \exp[-a_1 z] \cos\left(a_2 z - a_3 t\right) \mathbf{e}_y.$$

Express each of the following quantities in terms of symbols that appear in the expression for $\mathbf{E}_{\text{phys}}$ and the speed of light in free space c:

$k_{\text{real}},\quad k_{\text{imag}},\quad \omega,\quad n_{\text{real}},\quad n_{\text{imag}},\quad \varepsilon_{\text{real}},\quad \varepsilon_{\text{imag}},\quad v_{\text{phase}},\quad \text{absorption length}.$

Exercise 4.3 The relative permittivity of amorphous silica for infrared radiation with free space wavelength $2000\,\text{nm}$ is $1.46 + \mathrm{i}\,1.0 \times 10^{-10}$. Its relative permeability is 1.0.

(a) Determine the values of

(i) the refractive index,

(ii) the wavenumber,

(iii) the phase speed,

(iv) the factor by which the amplitude of a plane wave decreases over a distance of $1.0\,\text{km}$.

(b) Use the results of part (a) to write down expressions for the physical electric and magnetic fields of a plane wave with free space wavelength $2000\,\text{nm}$ travelling through amorphous silica in the x-direction, polarized in the y-direction. Assume that the amplitude of the electric field oscillation at the origin is E_0.

Exercise 4.4 The behaviour of a particular dielectric material is represented by the simple classical model described in Section 4.2 of Book 3, with plasma frequency ω_p, friction parameter γ and natural frequency ω_n related by

$$\omega_\text{p} = 2.00\omega_\text{n}, \quad \gamma = \omega_\text{n}, \quad \text{and} \quad \omega_\text{n} = 4.0\times 10^{15}\,\text{s}^{-1}.$$

Determine the values of the following quantities:

(a) the real and imaginary parts of the dielectric function in the limit $\omega \to 0$,

(b) the real and imaginary parts of the dielectric function in the limit $\omega \to \infty$,

(c) the angular frequency at which $\varepsilon_{\text{real}} = 1$,

(d) the value of $\varepsilon_{\text{imag}}$ at $\omega = \omega_n$. (This is approximately the angular frequency at which $\varepsilon_{\text{imag}}$ has its maximum value.)

Exercise 4.5 (a) A steak has relative permittivity $\varepsilon = 40 + \mathrm{i}\,12$ at $2.45\,\text{GHz}$, the frequency used in microwave ovens. Estimate the depth over which the amplitude of the electric field inside the steak falls to half of the external value.

(b) Polystyrene has $\varepsilon = 1.03 + \mathrm{i}\,3 \times 10^{-5}$ at $2.45\,\text{GHz}$. Explain why it is a suitable container for use in a microwave oven.

Exercise 4.6 Blue light is more strongly refracted by a glass prism than red light.

(a) What is the sign of $\mathrm{d}n/\mathrm{d}\omega$ for glass at optical frequencies?

(b) Is the group speed greater or less than the phase speed for light in glass?

(c) For a short pulse of light travelling through glass, would the individual maxima and minima of the electric field move from the rear to the front of the pulse or from the front to the rear?

Exercise 4.7 For many types of glass used to make lenses, the refractive index is well-represented by the empirical expression

$$n = A + \frac{B}{\lambda_0^2},$$

where λ_0 is the free space wavelength, and for hard crown glass (type K5) the values of the constants are $A = 1.522$ and $B = 4.59 \times 10^{-15}\,\mathrm{m}^2$.

Determine each of the following quantities for both blue light with free space wavelength $\lambda_{\mathrm{blue}} = 400\,\mathrm{nm}$ and red light with $\lambda_{\mathrm{red}} = 600\,\mathrm{nm}$:

(a) refractive index;

(b) phase speed;

(c) group speed;

(d) the transmission angle (angle of refraction) when light is incident on the glass surface in air at 30° to the normal.

(For the speed of light, use the more precise value than is usually used in this course, i.e. c, $2.998 \times 10^8\,\mathrm{m\,s^{-1}}$, .)

Exercise 4.8 Use data from Chapter 4 of Book 3 to estimate the absorption lengths in pure water and in amorphous silica for electromagnetic radiation with free space wavelengths of $500\,\mathrm{nm}$, $1000\,\mathrm{nm}$ and $2000\,\mathrm{nm}$.

Exercise 4.9 Pulses of light are launched into an optical fibre from a modulated light source that produces square pulses of duration $1\,\mu\mathrm{s}$, separated by $10\,\mu\mathrm{s}$. The source emits light in two narrow bands centred on free space wavelengths of $600\,\mathrm{nm}$ and $650\,\mathrm{nm}$ and with spreads of about $5\,\mathrm{nm}$ around these wavelengths. The group speed for light in the fibre is given by the expression

$$v_{\mathrm{group}} = c\left(A + \frac{B}{\lambda_0^2}\right)^{-1},$$

where λ_0 is the free space wavelength, and $A = 1.4580$ and $B = 1.062 \times 10^{-14}\,\mathrm{m}^2$.

For reliable communication, the separation of the two wavelength components of each pulse is required to be less than 10% of the $10\,\mu\mathrm{s}$ separation between successive pulses. What is the maximum length D of fibre that can be used if this condition is to be satisfied?

Book 3 Chapter 5

Exercise 5.1 A simple classical model predicts that the relative permittivity function for a conductor has the form

$$\varepsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + \mathrm{i}\,\omega/\tau_{\mathrm{c}}}.$$

(a) What is the significance of the terms ω_{p} and τ_{c}?

(b) What is the condition for $\varepsilon(\omega)$ to be real?

(c) What is the value of $\varepsilon(\omega)$ for very high frequencies, and what is the physical significance of this result?

(d) What can you say about the values of $\varepsilon(\omega)$ for frequencies in the range $1/\tau_{\mathrm{c}} \ll \omega < \omega_{\mathrm{p}}$, and what consequence does this have for the nature of electromagnetic disturbances in this range?

(e) In the low frequency limit, derive an expression for $\varepsilon(\omega)$ that involves the conductivity σ.

Exercise 5.2 Compare the forms of the electric and magnetic fields just below the plane surfaces of a dielectric material and a metallic conductor when an electromagnetic plane wave travelling through air is normally incident on the surfaces.

Exercise 5.3 A thin metallic strip on a circuit board has length L, width a and thickness t, with $L \gg a \gg t$. Derive an expression for the ratio of the resistance between the ends of the strip at frequency f to its DC resistance, assuming that the skin depth is small compared with the thickness t.

Exercise 5.4 How many TE modes with $\omega = 10\,\mathrm{GHz}$ can propagate between two parallel conducting planes separated by $10\,\mathrm{cm}$?

Exercise 5.5 Use Faraday's law to derive expressions for the magnetic field components that accompany the electric field given by the expression in Equation 5.34 of Book 3.

Exercise 5.6 Starting from the dispersion relation for the TE_{mn} mode in a rectangular waveguide with internal cross-section $a \times b$, with $a > b$, derive an expression for the group speed of this mode.

Exercise 5.7 A rectangular waveguide has internal cross-section with dimensions $3.4\,\mathrm{cm} \times 1.7\,\mathrm{cm}$.

(a) What are the lowest four cut-off frequencies for TE modes in this waveguide?

(b) Show that the phase speed of the TE_{mn} mode is given by

$$v_{\mathrm{phase}} = c\left(1 - \frac{f_{mn}^2}{f^2}\right)^{-1/2}.$$

and hence determine the phase speeds and groups speeds for the four modes identified in part (a) when $f = 10\,\mathrm{GHz}$.

Book 3 Chapter 6

Exercise 6.1 The number density of electrons in a collisionless plasma is $10^{10}\,\mathrm{m}^{-3}$. What are the values of the group speed and the phase speed in this plasma for electromagnetic waves with frequencies $1.00\,\mathrm{MHz}$, $2.00\,\mathrm{MHz}$ and $100\,\mathrm{MHz}$?

Exercise 6.2 A pulsar, which is 10^{19} m from Earth, emits pulses of electromagnetic radiation across a

broad spectrum. The pulses travel to Earth through the interstellar medium, which can be regarded as a dilute plasma within which the number density of electrons and ions is about $3 \times 10^4\,\mathrm{m^{-3}}$. What is the difference in arrival time at Earth of the pulse detected with a radio telescope sensitive to frequencies around $100\,\mathrm{MHz}$ and the corresponding pulse detected with an optical telescope with a red filter?

Exercise 6.3 An electromagnetic plane wave with angular frequency ω is travelling in the z-direction and is normally incident on a dielectric window in a chamber containing a collisionless isotropic plasma that has plasma frequency $\omega_\mathrm{p} = 2\pi \times 10^8\,\mathrm{s^{-1}}$. Derive an expression for the ratio of the amplitude of the electric field in the plasma at a distance z from the window to the amplitude just inside the window when $\omega < \omega_\mathrm{p}$, assuming that the field is constant across a plane with constant z within a 'beam' defined by the area of the window. Hence evaluate this ratio for a distance $z = 0.5\,\mathrm{m}$ for angular frequencies $0.99\omega_\mathrm{p},\ 0.90\omega_\mathrm{p}$ and $0.50\omega_\mathrm{p}$, and its limiting value when $\omega/\omega_\mathrm{p} \ll 1$.

Exercise 6.4 A plasma used for etching is heated by coupling $13.56\,\mathrm{MHz}$ radiowaves into the plasma in an applied magnetic field. The plasma frequency is $\omega_\mathrm{p} = 8.0 \times 10^{10}\,\mathrm{s^{-1}}$ and the cyclotron frequency is $\omega_\mathrm{c} = 3.0 \times 10^8\,\mathrm{s^{-1}}$.

(a) What are the values of the electron number density and the magnetic field strength in the plasma?

(b) What type of waves are produced in the plasma, and what is their group speed?

Exercise 6.5 Microwaves with wavelength $3.0\,\mathrm{cm}$ are to be used to heat a plasma.

(a) What steady magnetic field strength is required for electron cyclotron resonance to occur?

(b) Use information from Figures 6.7 and 6.9 of Book 3 to explain why it would not be efficient to heat a plasma that is at a pressure of 10^4 Pa with this microwave source.

Book 3 Chapter 7

Exercise 7.1 State whether the changes listed below would increase, decrease or leave unchanged the scattering cross-section of a collagen fibril embedded in the transparent matrix of the cornea, and give a reason to support each answer. Assume that each of the changes, apart from (f), is small.

(a) Increase in the fibril diameter.

(b) Increase in the refractive index of the fibril.

(c) Increase in the refractive index of the matrix.

(d) Increase in the spacing of the fibrils.

(e) Increase in the wavelength of the radiation.

(f) Changing the polarization of the incident radiation from parallel to the fibril to perpendicular to the fibril.

Exercise 7.2 The fundamental resonance frequency for a hairpin resonator in a vacuum is $f_0 = 500\,\mathrm{MHz}$. At what other frequencies might resonances be observed for this hairpin in a vacuum?

Exercise 7.3 A (long) hairpin resonator has its fundamental resonance frequency in vacuum at $f_0 = 100\,\mathrm{MHz}$. What will the fundamental resonance frequency be when the hairpin is immersed in (a) water $(\varepsilon = 80)$, (b) ethanol $(\varepsilon = 25)$ and (c) a cold collisionless plasma with electron number density $2.0 \times 10^{13}\,\mathrm{m^{-3}}$?

Exercise 7.4 The fundamental resonance frequency of a hairpin resonator, length $10.0\,\mathrm{mm}$, immersed in a dilute cold collisionless plasma is twice the value of the fundamental resonance frequency when the hairpin is in a vacuum. What is the number density of electrons in the plasma?

Solutions for Book 1

Book 1 Chapter 1

Solution 1.1 According to the suggested model, the person contains $10 \times 60\,\text{kg}/(3.0 \times 10^{-26}\,\text{kg})$ electrons, each of charge $-1.6 \times 10^{-19}\,\text{C}$, giving a total electronic charge of $-3.2 \times 10^{9}\,\text{C}$. This is normally exactly balanced by the positive charge of protons — so, to have a net charge of $1\,\mu\text{C}$, the person would need to lose a fraction $10^{-6}/3.2 \times 10^{9} = 3.1 \times 10^{-16}$ of their electrons.

Solution 1.2 The charge-to-mass ratio of a proton is

$$\frac{e}{m} = \frac{1.60 \times 10^{-19}\,\text{C}}{1.67 \times 10^{-27}\,\text{kg}} = 9.58 \times 10^{7}\,\text{C}\,\text{kg}^{-1}.$$

So one gram of protons has charge $q = 9.58 \times 10^{4}\,\text{C}$. Coulomb's law gives an electrostatic force of magnitude

$$\begin{aligned} F &= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2} \\ &= 8.99 \times 10^{9}\,\text{N}\,\text{m}^2\,\text{C}^{-2} \times \frac{(9.58 \times 10^{4}\,\text{C})^2}{(3.84 \times 10^{8}\,\text{m})^2} \\ &= 560\,\text{N}. \end{aligned}$$

This is a significant force, in spite of the enormous separation of the charges! Of course, it would not be easy to gather one gram of protons together in a small region — the repulsive forces would be huge.

Solution 1.3 The charge q experiences no force because it is midway between two identical charges. The two outer charges repel one another with a force of magnitude $Q^2/4\pi\varepsilon_0 d^2$. The charge q must provide a balancing attraction so

$$\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{d^2} + \frac{1}{4\pi\varepsilon_0}\frac{q\,Q}{(d/2)^2} = 0,$$

which gives $q = -Q/4$. The equilibrium is not stable. For example, if q is moved slightly to the left, the attraction from the left-hand charge exceeds that from the right-hand charge so q tends to move even further to the left rather than returning to its equilibrium position.

Solution 1.4 (a) Charges at opposite corners produce opposite electric fields at the centre of the square, so the total electric field is zero. (b) When one of the corner charges is removed, the pair of charges in opposite corners still produce opposite electric fields at the centre of the square and therefore cancel out. The third charge produces a field of magnitude

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{(d/\sqrt{2})^2} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{d^2}.$$

This is the total field at the centre of the square.

Solution 1.5 The problem can be simplified by noting that the two $2q$ charges produce opposite fields at the centre of the square, so their contributions cancel out. We are left with:

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(d/\sqrt{2})^2}\frac{\mathbf{e}_x - \mathbf{e}_y}{\sqrt{2}} + \frac{4q}{(d/\sqrt{2})^2}\frac{\mathbf{e}_x - \mathbf{e}_y}{\sqrt{2}}\right) \\ &= \frac{5\sqrt{2}}{4\pi\varepsilon_0}\frac{q}{d^2}\,(\mathbf{e}_x - \mathbf{e}_y). \end{aligned}$$

Solution 1.6 By symmetry, the electric field vanishes at the centre of the disk. The field due to any element of the disk is exactly cancelled by the field due a corresponding element on the opposite side of the disk.

Solution 1.7 (a) By symmetry, the total electric field vanishes at the centre of a uniformly-charged ring. The contribution from each small element of the ring is cancelled by the contribution from a similar element at the opposite end of the diameter passing through the centre of the ring.

(b) This part of the question is most easily answered using the principle of superposition. Let the electric field due to the complete ring be $\mathbf{E}_{\text{ring}}$, the electric field due to the removed segment be $\mathbf{E}_{\text{seg}}$ and the electric field due to the almost complete ring be $\mathbf{E}_{\text{rem}}$. Then the principle of superposition tells us that

$$\mathbf{E}_{\text{ring}} = \mathbf{E}_{\text{seg}} + \mathbf{E}_{\text{rem}}.$$

At the centre of the ring we have seen that $\mathbf{E}_{\text{ring}} = \mathbf{0}$, so

$$\mathbf{E}_{\text{rem}} = -\mathbf{E}_{\text{seg}}.$$

The charge of the segment is $Q\,l/2\pi R$ (the total charge of the ring times the fraction of the circumference of the ring occupied by the segment). If the segment is short, it behaves like a point charge. At the centre of the ring it produces an electric field

$$\mathbf{E}_{\text{seg}} = \frac{1}{4\pi\varepsilon_0}\frac{Q\,l/2\pi R}{R^2}\,\widehat{\mathbf{r}} = \frac{Q\,l}{8\pi^2\varepsilon_0 R^3}\,\widehat{\mathbf{r}},$$

where $\widehat{\mathbf{r}}$ is a unit vector pointing radially away from the segment. The almost complete ring produces the opposite electric field

$$\mathbf{E}_{\text{rem}} = -\frac{Q\,l}{8\pi^2\varepsilon_0 R^3}\,\widehat{\mathbf{r}},$$

which points radially *towards* the gap in the ring.

Solution 1.8 The magnitude of the field is

$$E = \sqrt{(3\,\text{N}\,\text{C}^{-1})^2 + (4\,\text{N}\,\text{C}^{-1})^2} = 5\,\text{N}\,\text{C}^{-1}$$

and its unit vector is

$$\widehat{\mathbf{E}} = \frac{(3\mathbf{e}_x + 4\mathbf{e}_y)\,\text{N}\,\text{C}^{-1}}{5\,\text{N}\,\text{C}^{-1}} = (0.6\mathbf{e}_x + 0.8\mathbf{e}_y).$$

The component of the field in the direction the unit vector $\widehat{\mathbf{u}}$ is given by the scalar product

$$E_u = \mathbf{E} \cdot \widehat{\mathbf{u}} = \frac{3\,\text{N}\,\text{C}^{-1}}{\sqrt{2}} - \frac{4\,\text{N}\,\text{C}^{-1}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\,\text{N}\,\text{C}^{-1}.$$

Book 1 Chapter 2

Solution 2.1 Because the charge distribution is cylindrically symmetric, we surround the z-axis by a nested set of thin cylindrical tubes, extending from $z = 0$ to $z = L$ and from $r = 0$ to $r = R$. A thin cylindrical tube with radii between r and $r + \Delta r$ has volume $2\pi r\,\Delta r \times L$, where Δr is the thickness of the tube. The charge density in this tube is A/r, so the tube contributes a charge

$$\Delta Q = \frac{A}{r} \times 2\pi r\,\Delta r \times L = 2\pi AL\,\Delta r.$$

The total charge Q is obtained by integrating over all the tubes from $r = 0$ to $r = R$, giving

$$Q = \int_0^R 2\pi AL\,\mathrm{d}r = 2\pi ALR.$$

Solution 2.2 (a) This situation is spherically symmetric, so we split the volume into thin spherical shells centred on the origin. The total charge of the electron cloud is

$$\begin{aligned} Q_{\text{elec}} &= \int_0^\infty A\exp(-2r/a_0) \times 4\pi r^2\,\mathrm{d}r \\ &= 4\pi A \int_0^\infty r^2 \exp(-2r/a_0)\,\mathrm{d}r. \end{aligned}$$

Changing the variable of integration to $x = 2r/a_0$ and using the standard integral given in the question, we obtain

$$Q_{\text{elec}} = 4\pi A \times \left(\frac{a_0}{2}\right)^3 \int_0^\infty x^2 \exp(-x)\,\mathrm{d}x = \pi a_0^3\,A.$$

In order for the atom to be neutral the charge of the electron cloud must be opposite to the charge of the proton. So $Q_{\text{elec}} = -e$ and $A = -e/\pi a_0^3$.

(b) The charge density of the electron cloud at the Bohr radius is therefore

$$\begin{aligned} \rho(a_0) &= \frac{-e \times \exp(-2)}{\pi a_0^3} \\ &= \frac{-1.6 \times 10^{-19}\,\mathrm{C} \times 0.1353}{\pi \times (5.29 \times 10^{-11}\,\mathrm{m})^3} \\ &= -4.7 \times 10^{10}\,\mathrm{C\,m^{-3}}. \end{aligned}$$

Solution 2.3 If the field were produced by an isolated point charge Q at the origin, Gauss's law tells us that the electric flux over the any sphere centred on the origin is Q/ε_0, which is independent of the radius of the sphere. Surround the origin by a sphere of radius R. The vector field $\mathbf{E}$ is perpendicular to the surface of the sphere and has a constant normal component $A\exp(-R/a)$ over this surface. The total flux of $\mathbf{E}$ over the surface is therefore $A\exp(-R/a) \times 4\pi R^2$. This clearly depends on the radius of the sphere, becoming negligible for very large spheres. This is inconsistent with Gauss's law.

This argument works no matter how the charge is moving because Gauss's law is valid under all circumstances. It does not rule out the possibility that the field is produced by an extended distribution of charge. We can imagine positive and negative clouds of charge centred on the origin, with the negative cloud tailing off more gradually than the positive cloud. The total charge enclosed by a sphere centred on the origin would then depend on the radius of the sphere, avoiding the conflict with Gauss's law.

Solution 2.4 The divergence theorem is true for all vector fields.

However, the flux of $\mathbf{F}$ is not generally equal to zero through a closed surface that does not enclose the origin because the field dies away so quickly that distant parts of the surface contribute virtually no (positive) flux, while parts of the surface close to the origin contribute a sizeable negative flux. As an example, imagine a closed surface consisting of two nested hemispheres, one with a small radius r, the other with a large radius $R >> r$, both centered on the origin, and a flat ring in the xy-plane that closes the gap between the hemispheres (see Figure 8). The ring contributes no flux, the hemisphere with radius R negligible flux, while the hemisphere with radius r contributes a sizeable negative flux.

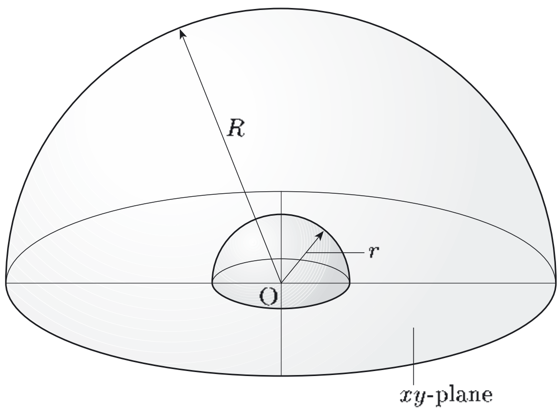

Figure 8 For Solution 2.4.

Solution 2.5 By symmetry, any electric field inside the cylindrical shell can only have a radial component and its magnitude can only depend on the distance r from the axis. In cylindrical polar coordinates,

$$\mathbf{E}(r, \phi, z) = E_r(r)\,\mathbf{e}_r.$$

Choose a cylindrical Gaussian surface, radius r, length δl, concentric with the cylindrical shell, but of smaller radius. No charge is enclosed by this surface so Gauss's law gives

$$E_r(r) \times 2\pi r \times \Delta l = 0$$

Thus $E_r(r) = 0$ for all $r \neq 0$. This argument shows that $\mathbf{E} = \mathbf{0}$ at all points inside the cylinder, except possibly along the central axis. However, the electric field also vanishes along this axis for symmetry reasons.

Solution 2.6 Let the point P at which the field is measured be a distance R from the centre of the cylinder (see Figure 9). Consider two symmetrically-placed positive charges Q in the distribution, each a distance r from the central point, where $r < R$. According to Coulomb's law and the principle of superposition, the magnitude of the field at P due to these two charges is

$$\begin{aligned} E_1 &= \frac{Q}{4\pi\varepsilon_0}\frac{1}{(R-r)^2} + \frac{Q}{4\pi\varepsilon_0}\frac{1}{(R+r)^2} \\ &= \frac{Q}{4\pi\varepsilon_0} \times \frac{(R+r)^2 + (R-r)^2}{(R-r)^2\,(R+r)^2} \\ &= \frac{2Q}{4\pi\varepsilon_0} \times \frac{R^2+r^2}{(R^2-r^2)^2}. \end{aligned}$$

For comparison, if the two charges were moved to the centre of the cylinder, the magnitude of the field at P would be

$$E_2 = \frac{2Q}{4\pi\varepsilon_0} \times \frac{1}{R^2}.$$

Now, it is easy to see that

$$\frac{R^2+r^2}{(R^2-r^2)^2} > \frac{R^2-r^2}{(R^2-r^2)^2} = \frac{1}{(R^2-r^2)} > \frac{1}{R^2}$$

so $E_1 > E_2$. Since this is true for all pairs of charges symmetrically-placed on either side of the centre of the cylinder, collapsing the entire distribution down to the central point would reduce the magnitude of the field.

This calculation does not use Gauss's law, but it shows that the results obtained from Gauss's law for spheres and cylinders that are collapsed radially are rather special.

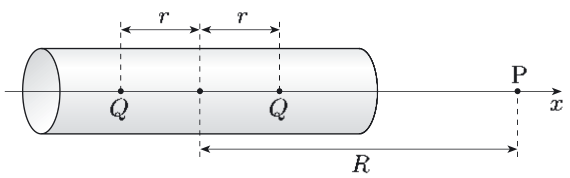

Figure 9 For Solution 2.6.

Solution 2.7 No, the statement is wrong. Obviously the electric field at any point depends in principle on all charges, no matter where they might be. Although charges outside the spherical surface do not contribute to the total flux over the surface, they do influence the electric field at individual points on the surface. It is just that they contribute a positive flux in some patches and a negative flux in others, and all these contributions cancel out when we integrate over the whole surface.

Solution 2.8 No, this is not possible. For example, it will not help to surround the charged cube by a cubic Gaussian surface because the electric field produced by the charged cube is not perpendicular to the faces of this Gaussian surface and is not constant over them. No other simple Gaussian surface is any better. We generally need spherical, cylindrical or planar symmetry to have any chance of obtaining complete knowledge of the electric field from the integral version of Gauss's law.

Solution 2.9 (a) The thickness of the membrane is much less than the radius of the cell, so we model it by a pair of plane sheets of charge. Inside the membrane, we can use the formula for the field between the plates of a parallel plate capacitor:

$$\begin{aligned} E &= \frac{\sigma}{\varepsilon_0} \\ &= \frac{2.5 \times 10^{-6}\,\mathrm{C\,m^{-2}}}{8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}} \\ &= 2.8 \times 10^5\,\mathrm{N\,C^{-1}}. \end{aligned}$$

(b) The membrane provides two spherical shells of charge. A uniformly-charged spherical shell produces no electric field in the spherical volume it surrounds, so there is no electric field in the cell.

(c) Exploiting spherical symmetry we choose a Gaussian surface concentric with the cell and of greater radius than the cell. This surface contains zero total charge so Gauss's law tells us that the electric flux is zero over the surface. By spherical symmetry, the electric field is everywhere radial and has the same magnitude at all points on the spherical surface. We conclude that the electric field vanishes outside the cell.

Solution 2.10 (a) By symmetry, the electric field is aligned with the z-axis, $\mathbf{E} = E(z)\,\mathbf{e}_z$. For $z > 0$ it is in the positive z-direction and for $z < 0$, it is in the negative z-direction.

To determine the electric field outside the slab, we choose a cylindrical Gaussian surface of cross-section ΔA with end-faces at z and $-z$ outside the slab. Then Gauss's law gives

$$E(z)\,\Delta A + E(-z)\,\Delta A = \frac{\rho\,\Delta A\,d}{\varepsilon_0},$$

where E is the *magnitude* of the electric field. By symmetry, $E(z) = E(-z)$, so

$$E(z) = \frac{\rho d}{2\varepsilon_0}.$$

So the electric field is

$$\mathbf{E} = \begin{cases} \dfrac{\rho d}{2\varepsilon_0}\,\mathbf{e}_z & \text{for} \quad z > \frac{d}{2}, \\[2ex] -\dfrac{\rho d}{2\varepsilon_0}\,\mathbf{e}_z & \text{for} \quad z < \frac{d}{2}. \end{cases}$$

To determine the electric field inside the slab we choose a similar cylindrical Gaussian surface of cross-section ΔA, but with end-faces at z and $-z$ inside the slab. This time Gauss's law gives

$$E(z)\,\Delta A + E(-z)\,\Delta A = \frac{\rho\,\Delta A \times 2z}{\varepsilon_0}.$$

Following through a similar calculation to that given above, we conclude that

$$\mathbf{E} = \frac{\rho z}{\varepsilon_0}\,\mathbf{e}_z \quad \text{for} \quad -\frac{d}{2} < z < \frac{d}{2}.$$

(b) In both regions outside the slab, the electric field is constant and therefore has zero divergence. This agrees with the differential version of Gauss's law because there is no charge density outside the slab. Evaluating the divergence inside the slab we obtain

$$\operatorname{div}\mathbf{E} = \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0},$$

which is consistent with the differential version of Gauss's law. The surfaces of the slab create no problems because the electric field is continuous there.

Solution 2.11 We solve for the plane and the slab separately and then use the principle of superposition. Using Gauss's law we can find the electric fields due to the plane and slab everywhere. The calculations are not given here (they essentially repeat the working of the previous problem) but the results are as follows.

The field due to the slab is

$$E_z^{\text{slab}} = \begin{cases} \dfrac{\rho d}{2\varepsilon_0} & \text{for } z > d/2, \\ \dfrac{\rho z}{\varepsilon_0} & \text{for } -d/2 < z < d/2, \\ -\dfrac{\rho d}{2\varepsilon_0} & \text{for } z < -d/2. \end{cases}$$

The field due to the plane is

$$E_z^{\text{plane}} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0} & \text{for } z > d/2, \\ -\dfrac{\sigma}{2\varepsilon_0} & \text{for } z < d/2. \end{cases}$$

Using the principle of superposition, the total field is

$$E_z = E_z^{\text{slab}} + E_z^{\text{plane}} = \begin{cases} \dfrac{\rho d + \sigma}{2\varepsilon_0} & \text{for } z > \frac{d}{2}, \\ \dfrac{2\rho z - \sigma}{2\varepsilon_0} & \text{for } -\frac{d}{2} < z < \frac{d}{2}, \\ -\dfrac{\rho d + \sigma}{2\varepsilon_0} & \text{for } z < -\frac{d}{2}. \end{cases}$$

This vanishes inside the slab at $z = \sigma/2\rho$, provided that $-d/2 < \sigma/2\rho < d/2$.

Solution 2.12 The negative charge within the Bohr radius is

$$Q = \int_0^{a_0} \rho \times 4\pi r^2\,\mathrm{d}r = -\frac{4e}{a_0^3}\int_0^{a_0} r^2 \exp(-2r/a_0)\,\mathrm{d}r.$$

Changing the variable of integration to $x = 2r/a_0$ we obtain

$$Q = -\frac{4e}{a_0^3} \times \left(\frac{a_0}{2}\right)^3 \int_0^2 x^2 \exp(-x)\,\mathrm{d}x$$
$$= -\frac{e}{2} \times 0.647 = -0.32e.$$

Allowing for the positive point charge at the centre, the net charge within the Bohr radius is $0.68e$.

Let the radial component of the electric field at a_0 be E_r. Then Gauss's law gives

$$4\pi a_0^2 \times E_r = \frac{0.68e}{\varepsilon_0},$$

so

$$E_r = \frac{0.68e}{4\pi\varepsilon_0 a_0^2} =$$
$$\frac{0.68 \times 1.6 \times 10^{-19}\,\mathrm{C}}{4\pi \times 8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}} \times (5.29 \times 10^{-11}\,\mathrm{m})^2}$$
$$= 3.5 \times 10^{11}\,\mathrm{N\,C^{-1}}.$$

Solution 2.13 It is helpful to add a second hemispherical shell, to create a complete uniformly-charged spherical shell. Because the charge distribution is spherically symmetric, the electric field inside the complete sphere is zero. Let the electric field due to the first hemispherical shell be $\mathbf{E}_1$ and the electric field due to the second hemispherical shell be $\mathbf{E}_2$. Then, inside the sphere we have

$$\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{0}. \qquad (*)$$

The second hemispherical shell is obtained by reflecting the first hemispherical shell across the imaginary surface S. It follows that the field pattern produced by the second hemispherical shell is the reflection of the field pattern produced by the first hemispherical shell. We have

$$E_{1x} = E_{2x} \quad \text{and} \quad E_{1y} = E_{2y} \quad \text{while} \quad E_{1z} = -E_{2z}$$

Combining this with Equation ($*$), we see that the x- and y-components vanish. It follows that the electric field due to a single hemisphere is perpendicular to S.

Solution 2.14 (a) From Gauss's law, the total electric flux over the cube is Q/ε_0. Because the charge is at the centre of the cube, each of its 6 faces contributes equally to the flux. The flux over a single face is $Q/6\varepsilon_0$.

(b) Faces of the cube can be split into two groups: three "adjacent" faces (adjacent to the corner where the charge is located) and three "opposite" faces (opposite to the corner where the charge is located). By symmetry, the three "adjacent" faces have identical fluxes and the three "opposite" faces have identical fluxes. (See Figure 10).

Imagine now 8 cubes, identical to the one considered, arranged to form a larger cube, with double the side length, in such a way that the point charge Q is near the centre of the larger cube. The flux over the entire larger cube is Q/ε_0 and the flux over each of its faces is $Q/6\varepsilon_0$. One of the faces of the larger cube can be split into four equal squares with equal fluxes, one of

which is an "opposite" face from the original cube. We conclude that the flux over each "opposite" face is

$$\frac{1}{4}\,\frac{Q}{6\varepsilon_0} = \frac{Q}{24\varepsilon_0},$$

leaving each "adjacent" face to have a flux

$$\frac{1}{3}\left(\frac{Q}{\varepsilon_0} - \frac{3Q}{24\varepsilon_0}\right) = \frac{7Q}{24\varepsilon_0}.$$

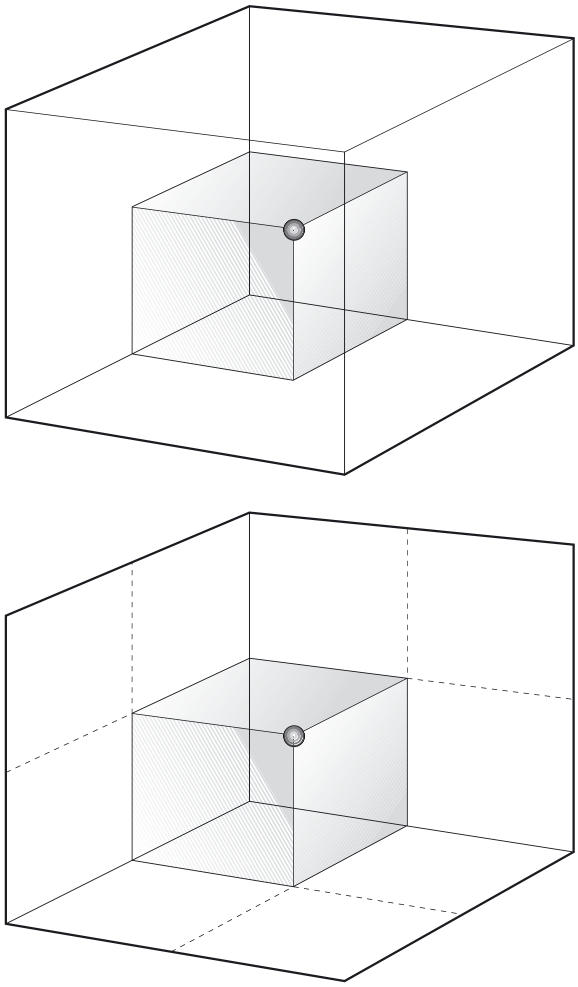

Figure 10 For Solution 2.14b.

Solution 2.15 Because the shells are conducting, the charge spreads out evenly over each shell, ensuring a spherically symmetric charge distribution and therefore a spherically symmetric field. The electric field outside any spherically symmetric charge distribution is as if all the charge were concentrated at the central point. Using this fact, the field outside both spheres points radially inwards when $q_1 + q_2 < 0$, is zero when $q_1 + q_2 = 0$ and points radially outwards when $q_1 + q_2 > 0$.

Solution 2.16 Suppose we wish to trap a positive ion. In order for a restoring force to act on this ion the electric field must point inwards at every point on the boundary of the trapping region. So the electric flux over the surface of the trapping region is negative. Gauss's law then requires there to be a negative charge in the trapping region. This contradicts our initial assumption that the trapping region is a vacuum containing no charges, apart from the ion. The positive charge of the ion does not help here because (a) it has the wrong sign and (b) it is irrelevant because we are looking for the origins of a field that exerts a force on the ion. The ion cannot exert a force on itself.

The fact that stable equilibrium cannot be achieved with externally-imposed static electric fields alone is known as **Earnshaw's theorem**. Its practical consequence is that an ion trap cannot be constructed using electrostatics alone; successful ion traps use magnetic fields.

Solution 2.17 According to Gauss's law

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}.$$

The partial derivatives $\partial E_y/\partial y$ and $\partial E_z/\partial z$ are equal to zero because the field has no y- or z-components. The partial derivative $\partial E_x/\partial x$ can be written as an ordinary derivative because E_x depends only on x. Hence Gauss's law becomes

$$\frac{\mathrm{d}E_x}{\mathrm{d}x} = \frac{\rho_0}{\varepsilon_0}\cos(kx).\ (\text{See Figure 11}).$$

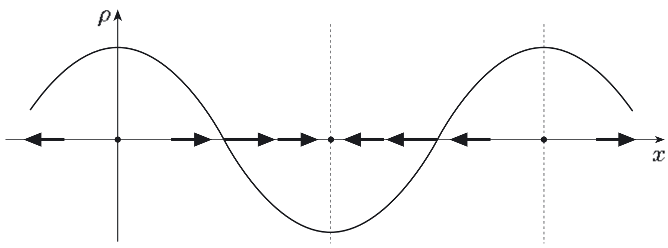

Figure 11 For Solution 2.17.

Integrating both sides with respect to x gives

$$E_x = \frac{\rho_0}{\varepsilon_0 k}\sin(kx) + C,$$

where C is an arbitrary constant. Using the initial value $E_x = 0$ at $x = 0$ we see that $C = 0$, so we conclude that

$$\mathbf{E} = \frac{\rho_0}{\varepsilon_0 k}\sin(kx)\,\mathbf{e}_x.$$

Book 1 Chapter 3

Solution 3.1 A current I flowing in a loop of cross-sectional area A produces a magnetic moment of magnitude $m = |I|A$. Thus

$$|I| = \frac{m}{\pi R^2} = \frac{8.0 \times 10^{22}\,\mathrm{A\,m^2}}{\pi \times (2.35 \times 10^6\,\mathrm{m})^2} = 4.6 \times 10^9\,\mathrm{A}.$$

Solution 3.2 The current flow is in the direction of **J**, which is in the direction of **A**. If the current density is to produce no accumulation of charge, its

divergence must vanish. We have

$$\begin{aligned}\frac{\partial J_x}{\partial x} &= A_x \frac{\partial \exp(-\mathbf{k}\cdot\mathbf{r})}{\partial x}\\ &= A_x \frac{\partial \exp(-(k_x x + k_y y + k_z z))}{\partial x}\\ &= -k_x A_x \exp(-\mathbf{k}\cdot\mathbf{r}),\end{aligned}$$

with similar results for the y- and z-components. So,

$$\operatorname{div}\mathbf{J} = -(\mathbf{k}\cdot\mathbf{A})\exp(-\mathbf{k}\cdot\mathbf{r}).$$

The required condition for $\operatorname{div}\mathbf{J}$ to vanish is therefore that $\mathbf{k}$ is orthogonal to $\mathbf{A}$.

Solution 3.3 Consider a current loop in the plane of the page, with the current in an anticlockwise direction viewed from above. The Biot–Savart law gives the magnetic force on current element $I\,\delta\mathbf{l}_1$ due to current element $I\,\delta\mathbf{l}_2$ and can be written as

$$\mathbf{F_{12}} = \frac{\mu_0}{4\pi}\,\frac{I\,\delta\mathbf{l}_1 \times (I\,\delta\mathbf{l}_2 \times \widehat{\mathbf{r}}_{12})}{r_{12}^2}.$$

One application of the right-hand rule shows that $I\,\delta\mathbf{l}_2 \times \widehat{\mathbf{r}}_{12}$ points out of the page towards you. This is true for any current element $I\,\delta\mathbf{l}_2$ that does not coincide with $I\,\delta\mathbf{l}_1$. A second application of the right-hand rule shows that $I\,\delta\mathbf{l}_1 \times (I\,\delta\mathbf{l}_2 \times \widehat{\mathbf{r}}_{12})$ is in the plane of the loop and points radially outwards from its centre. Adding up contributions from all current elements $I\,\delta\mathbf{l}_2$ we conclude that the total magnetic force on $I\,\delta\mathbf{l}_1$ is in the plane of the loop and points radially outwards.

Solution 3.4 This question refers back to Gauss's law. Consider a Gaussian surface surrounding the empty trapping region. There is no charge inside it, so the total electric flux over the surface is zero. Suppose the electric field points towards the origin along the z-axis. Then parts of the Gaussian surface near the z-axis contribute negative electric flux. To compensate for this, other parts of the Gaussian surface must contribute positive electric flux; this implies that the electric field points outwards, away from the origin, in at least one direction perpendicular to the z-axis. So, if the electric field confines along the axis of the magnetic field, it opposes confinement in a direction perpendicular to the field. In practice, this means that the electric field must not be too large.

Book 1 Chapter 4

Solution 4.1 The spherical symmetry suggests the use of spherical coordinates. Spherical symmetry shows that the spherical components of the magnetic field do not depend on the angular coordinates θ and ϕ. So,

$$\mathbf{B}(\mathbf{r}) = B_r(r)\,\mathbf{e}_r + B_\phi(r)\,\mathbf{e}_\phi + B_\theta(r)\,\mathbf{e}_\theta.$$

It is possible to find $180°$ rotations through the centre of the current distribution that leave the current distribution unchanged but reverse B_ϕ and B_θ. Symmetry then requires that

$$\begin{aligned}&B_r(r)\,\mathbf{e}_r + B_\phi(r)\,\mathbf{e}_\phi + B_\theta(r)\,\mathbf{e}_\theta\\ &\quad = B_r(r)\,\mathbf{e}_r - B_\phi(r)\,\mathbf{e}_\phi - B_\theta(r)\,\mathbf{e}_\theta,\end{aligned}$$

so the angular components vanish, leaving a radial magnetic field

$$\mathbf{B}(\mathbf{r}) = B_r(r)\,\mathbf{e}_r.$$

The no-monopole equation tells us that the magnetic flux vanishes over any closed surface. Evaluating the magnetic flux over a spherical surface S of radius R, centred on the origin, gives

$$\int_S \mathbf{B}\cdot\mathrm{d}\mathbf{S} = B(R)\times 4\pi R^2,$$

so $B(R) = 0$ for all $R > 0$. The magnetic field also vanishes at the origin because spherical symmetry prevents it from pointing in one direction rather than another.

Solution 4.2 It is sensible to express the field in cylindrical coordinates, with the z-axis perpendicular to the ring, and passing through its centre. Axial symmetry ensures that the cylindrical coordinates of the magnetic field do not depend on ϕ, so

$$\mathbf{B}(\mathbf{r}) = B_r(r,z)\,\mathbf{e}_r + B_\phi(r,z)\,\mathbf{e}_\phi + B_z(r,z)\,\mathbf{e}_z.$$

Consider time reversal, followed by a reflection in a plane that contains the z-axis. The net effect of these two operations is to leave the current distribution unchanged. Time reversal reverses all components of the magnetic field. Using the reflection rule, the reflection reverses $B_r(r,z)$ and $B_z(r,z)$ at any point in the plane of reflection (since these components are parallel to the plane), but $B_\phi(r,z)$ is unchanged (since this component is perpendicular to the plane). So the net effect of both operations is to convert $B_\phi(r,z)$ to $-B_\phi(r,z)$. Hence the ϕ-component of the magnetic field vanishes and the magnetic field therefore takes the form

$$\mathbf{B}(\mathbf{r}) = B_r(r,z)\,\mathbf{e}_r + B_z(r,z)\,\mathbf{e}_z.$$

Note that a reflection in the plane of the ring is useless in this problem because it changes z to $-z$. This prevents us from comparing two expressions for the magnetic field at a fixed point.

Solution 4.3 Because the situation is axially symmetric, we use cylindrical polar coordinates and express the magnetic field as

$$\mathbf{B} = B_\phi(r)\mathbf{e}_\phi,$$

where $B_\phi(r)$ is a function that remains to be determined.

Let R be the radius of the coaxial cable. Consider a circular path C of radius $r > R$, centred on, and perpendicular to, the central axis of the cable. The line integral of the magnetic field around this path is

$B_\phi(r) \times 2\pi r$. The circle C is the perimeter of a disk that covers the whole cross-section of the cable. Because the current carried by the inner cylinder is equal in magnitude and opposite in direction to the current carried by the outer tube, the total current flowing through this disk is zero. Consequently, the integral form of Ampère's law gives

$$B_\phi(r) \times 2\pi r = 0 \qquad \text{for all } r > R.$$

We conclude that the magnetic field vanishes everywhere outside the coaxial cable.

Solution 4.4 We can again use symmetry to argue that the magnetic field has no x- or z-components. We choose now a rectangular closed path in the plane $x = 0$, defined by the points A at $(x, y, z) = (0, 0, d)$, B at $(0, L, d)$, C at $(0, L, 0)$, and D at $(0, 0, 0)$. Exercises 4.5 and 4.10 in Book 1 showed that the magnetic field of an infinite current sheet does not fall off with distance from the sheet. Using this fact, together with the right-hand rule, we conclude that there is no magnetic field along AB — contributions from the two current sheets cancel out. Let the magnitude of the magnetic field along CD be B. Then Ampère's law gives

$$BL = \mu_0 I \times \frac{N}{l} \times L,$$

so, between the current sheets, the magnetic field has magnitude

$$B = \frac{\mu_0 IN}{l}.$$

Not surprisingly, this is twice as large as the magnetic field near a single current sheet. Indeed, it would be fine to use the principle of superposition, and the right-hand rule, to arrive directly at this answer.

Book 1 Chapter 5

Solution 5.1 Imagine building up the sphere in a series of spherical shells. Suppose we already have a sphere of uniform charge density ρ and radius r and we add a spherical shell of uniform charge density ρ and thickness δr. The thickness δr is taken to be small enough for us to neglect repulsion of elements within the spherical shell. The sphere has total charge $4\pi r^3\rho/3$ and the shell has total charge $4\pi r^2\,\delta r\,\rho$. The electric field of the sphere is just as if all its charge were concentrated at its centre, so the potential energy between these two charge distributions is

$$\delta U = \frac{1}{4\pi\varepsilon_0 r} \times \frac{4\pi r^3\rho}{3} \times 4\pi r^2\,\delta r\,\rho.$$

The total potential energy is found by integrating from 0 to the full radius R of the sphere:

$$U = \frac{1}{4\pi\varepsilon_0} \times \frac{(4\pi\rho)^2}{3} \int_0^R r^4\,\mathrm{d}r.$$

The total charge of the sphere is $Q = 4\pi R^3\rho/3$ so

$$U = \frac{3}{5}\,\frac{Q^2}{4\pi\varepsilon_0 R}.$$

Solution 5.2 At the centre of the sphere, a thin spherical shell of radius r, thickness δr and charge density ρ contributes a potential

$$\delta V = \frac{1}{4\pi\varepsilon_0 r}\,\rho \times 4\pi r^2\delta r.$$

The total potential at the centre due to the whole sphere is

$$V = \frac{\rho}{\varepsilon_0}\int_0^R r\,\mathrm{d}r = \frac{\rho R^2}{2\varepsilon_0}.$$

Since $Q = 4\pi R^3\rho/3$, we conclude that

$$V = \frac{3}{2}\,\frac{Q}{4\pi\varepsilon_0 R}.$$

An alternative method can be used. We can explicitly calculate the work done transporting a charge q radially from infinity to the centre of the sphere. Outside the sphere, the electric field is

$$\mathbf{E}_{\text{out}} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\mathbf{e}_r$$

so the work done bringing the charge from infinity to the surface of the sphere is

$$W_{\text{out}} = -\frac{qQ}{4\pi\varepsilon_0}\int_\infty^R \frac{1}{r^2}\,\mathrm{d}r = \frac{qQ}{4\pi\varepsilon_0 R}.$$

The electric field inside the sphere was derived in Worked Example 2.2 (Book 1, pp. 48-9), and is given by

$$\mathbf{E}_{\text{in}} = \frac{Qr}{4\pi\varepsilon_0 R^3}\,\mathbf{e}_r$$

so the work done bringing the charge from the surface of the sphere to the centre of the sphere is

$$W_{\text{in}} = -\frac{qQ}{4\pi\varepsilon_0 R^3}\int_R^0 r\,\mathrm{d}r = \frac{1}{2}\,\frac{qQ}{4\pi\varepsilon_0 R}.$$

The total work done is therefore

$$W = W_{\text{out}} + W_{\text{in}} = \frac{3}{2}\,\frac{qQ}{4\pi\varepsilon_0 R},$$

and the potential is

$$V = \frac{W}{q} = \frac{3}{2}\,\frac{Q}{4\pi\varepsilon_0 R}.$$

Note that there is no simple relationship between the potential energy of the whole sphere, which was calculated in Exercise 5.1, and the potential at the centre of the sphere. In general, potential energy and potential are very different concepts.

Solution 5.3 At the centre of the disk, a thin ring of radius r, thickness δr and surface charge density σ contributes a potential

$$\delta V = \frac{1}{4\pi\varepsilon_0 r}\,\sigma \times 2\pi r\,\delta r.$$

The total potential at the centre due to the whole disk is

$$V = \frac{\sigma}{2\varepsilon_0}\int_0^R \mathrm{d}r = \frac{\sigma R}{2\varepsilon_0}.$$

Since $Q = \pi R^2\sigma$, we conclude that

$$V = \frac{Q}{2\pi\varepsilon_0 R}.$$

Solution 5.4 The initial electrostatic energy is

$$U_1 = \frac{3}{5}\,\frac{Q^2}{4\pi\varepsilon_0 R}$$

(see Exercise 5.1). The final electrostatic energy is

$$U_2 = 2\times\frac{3}{5}\,\frac{(Q/2)^2}{4\pi\varepsilon_0\,(R/2^{1/3})} = 2^{-2/3}\,U_1 = 0.63\,U_1.$$

The change in electrostatic energy is

$$U_2 - U_1 = -0.37\,U_1.$$

Solution 5.5 The cloud and earth can be modelled as a large parallel plate capacitor. The electrostatic energy stored in the capacitor is

$$U = \frac{1}{2}\,qV = \frac{1}{2}\,30\,\mathrm{C}\times 1.0\times 10^9\,\mathrm{V} = 1.5\times 10^{10}\,\mathrm{J}.$$

This potential energy goes into light, sound, heat and the potential energy of ionized atoms.

Solution 5.6 In equilibrium, components of the electric field parallel to the surface of the conductor must vanish. If they did not vanish, free charges would flow in the conductor, contradicting our assumption of equilibrium. Since the electric field has no components parallel to the surface of the conductor, it must be perpendicular to the surface of the conductor.

Alternatively, we can argue that, in equilibrium, the surface of the conductor is an equipotential surface. The electric field is always perpendicular to an equipotential surface.

Book 1 Chapter 6

Solution 6.1 This problem can be tackled using either the magnetic force law or Faraday's law. We will consider both methods in turn.

Using the magnetic force law. A charge q in side AB experiences a magnetic field $B_z(x)$ and a magnetic force with y-component $F_y = -qv_xB_z(x)$. As it moves from A to B, the charge is displaced by $\Delta y = -L_1$ and the work done by the magnetic field is $F_y\,\Delta y = qv_xB_z(x)\times L_1$.

Similarly, a charge q in side CD experiences a magnetic field $B_z(x+L_2)$ and a magnetic force with y-component $F_y = -qv_xB_z(x+L_2)$. As it moves from C to D, the charge is displaced by $\Delta y = +L_1$ and the work done by the magnetic field is $F_y\,\Delta y = -qv_xB_z(x+L_2)\times L_1$.

These are the only electromagnetic forces that do work on the charge as it moves round the loop ABCD, so the total electromagnetic work done is

$$W = qv_xL_1[B_z(x) - B_z(x+L_2)]$$

and the emf is

$$V_{\mathrm{emf}} = \frac{W}{q} = v_xL_1[B_z(x) - B_z(x+L_2)].$$

Using Faraday's law. As the loop moves, the flux through it changes because the leading edge, CD, gathers more flux while the trailing edge, AB, leaves flux behind. If the loop moves a small distance Δx, the flux change associated with displacement of the leading edge is $B_z(x+L_2)\times L_1\,\Delta x$ while the flux change associated with displacement of the trailing edge is $-B_z(x)\times L_1\,\Delta x$. In a small time interval Δt, the loop moves a distance $\Delta x = v_x\,\Delta t$ and the total change in flux is

$$\Delta\Phi = [B_z(x+L_2) - B_z(x)]\times L_1v_x\,\Delta t.$$

The rate of change of flux is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = v_xL_1[B_z(x+L_2) - B_z(x)]$$

and Faraday's law gives

$$V_{\mathrm{emf}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = v_xL_1[B_z(x) - B_z(x+L_2)].$$

as before.

Solution 6.2 No emf is induced. The magnetic flux through the loop is constant, so Faraday's law tells us that the induced emf is zero.

Solution 6.3 Consider a positive charge q in the rod, at a distance r from the pivoted end. This charge has speed $v = r\omega$ and since its motion is perpendicular to the magnetic field it experiences a magnetic force of magnitude

$$F = qvB = qr\omega B.$$

Using the right-hand rule shows that this force acts radially outwards along the rod. If a charge q is transported from the pivot to the free end of the rod, the work done by the magnetic force is

$$W = \int_0^L F\,\mathrm{d}r = q\omega B\int_0^L r\,\mathrm{d}r = \tfrac{1}{2}q\omega BL^2,$$

and the voltage drop between the two ends of the rod is

$$V_{\mathrm{drop}} = \frac{W}{q} = \tfrac{1}{2}B\omega L^2.$$

This is the voltage drop as we move from the pivot to the free end of the rod.

Solution 6.4 As the rod rotates it sweeps out a circular sector OAB, where O is the pivot, OA is the initial position of the rod and OB is its position at a later time.

If θ is the angle between OA and OB, so that $\dot{\theta} = \omega$, the area of the sector OAB is

$$A = \frac{\theta}{2\pi} \times \pi L^2 = \tfrac{1}{2}\,\theta L^2.$$

The magnetic flux passing through the sector is

$$\Phi = \tfrac{1}{2}\,\theta L^2 B.$$

and Faraday's law gives

$$V_{\text{emf}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\tfrac{1}{2}\,\omega L^2 B.$$

This is the emf around the circuit OAB. The entire voltage drop takes place within the moving rod, this is the change in voltage as we move from the free end of the rod to the pivot, so it is minus the voltage drop calculated in the previous exercise.

Book 1 Chapter 7

Solution 7.1 Combining the equation of continuity with $\mathbf{J} = \sigma\mathbf{E}$, we obtain

$$\frac{\partial\rho}{\partial t} = -\operatorname{div}\mathbf{J} = -\operatorname{div}(\sigma\mathbf{E}).$$

The conductivity σ is a constant, so it can be moved outside the divergence. We can then use Gauss's law, $\operatorname{div}\mathbf{E} = \rho/\varepsilon_0$, to obtain

$$\frac{\partial\rho}{\partial t} = -\frac{\sigma}{\varepsilon_0}\,\rho.$$

At a fixed point in space, this differential equation has the solution

$$\rho = \rho_0\,\mathrm{e}^{-\sigma t/\varepsilon_0} = \rho_0\,\mathrm{e}^{-t/\tau},$$

where ρ_0 is a constant (the charge density at $t = 0$) and $\tau = \varepsilon_0/\sigma$ is the characteristic time constant of the exponential decay.

Comment: To get a feeling for the timescales involved, let's insert a typical value for the conductivity of distilled water ($\simeq 10^{-4}\,\Omega^{-1}\,\mathrm{m}^{-1}$) in our expression for τ. This gives

$$\tau = \frac{8.85 \times 10^{-12}\,\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}}{10^{-4}\,\Omega^{-1}\,\mathrm{m}^{-1}} \simeq 10^{-7}\,\mathrm{s}.$$

So any excess charge is dispersed very rapidly. In metals, charge dispersal is even faster, and for most purposes can be taken to be instantaneous.

Solution 7.2 Gauss's law and the no-monopole equation are satisfied because

$$\operatorname{div}\mathbf{E} = \frac{\partial}{\partial x} f(z - ct) = 0,$$

$$\operatorname{div}\mathbf{B} = \frac{1}{c}\frac{\partial}{\partial y} f(z - ct) = 0,$$

Faraday's law gives

$$\begin{aligned}\mathbf{0} &= \operatorname{curl}\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t}\\ &= \frac{\partial f}{\partial z}\mathbf{e}_y + \frac{1}{c}\frac{\partial f}{\partial t}\mathbf{e}_y\\ &= \left(\frac{\partial f}{\partial z} + \frac{1}{c}\frac{\partial f}{\partial t}\right)\mathbf{e}_y,\end{aligned}$$

and the Ampère–Maxwell law gives

$$\begin{aligned}\mathbf{0} &= \operatorname{curl}\mathbf{B} - \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\\ &= -\frac{1}{c}\frac{\partial f}{\partial z}\mathbf{e}_x - \frac{1}{c^2}\frac{\partial f}{\partial t}\mathbf{e}_x\\ &= -\frac{1}{c}\left(\frac{\partial f}{\partial z} + \frac{1}{c}\frac{\partial f}{\partial t}\right)\mathbf{e}_x.\end{aligned}$$

Both these equations are satisfied because, letting $u = z - ct$, we have

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} = \frac{\partial f}{\partial u}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = -c\frac{\partial f}{\partial u}.$$

Hence

$$\frac{\partial f}{\partial z} = -\frac{1}{c}\frac{\partial f}{\partial t}.$$

Comment: These electric and magnetic fields remain undisturbed in shape and travel in the positive z-direction at the constant speed c. This shows that electromagnetic disturbances need not be sinusoidal. In fact, any shape travelling with constant speed c can be represented as a linear combination of sinusoidal waves (using a mathematical technique called *Fourier analysis*). The monochromatic plane waves considered in the main text can therefore be thought of as basic building blocks from which all other electromagnetic waves can be made.

Solution 7.3 The electromagnetic wave is a plane wave, so the electric field strength is constant in any plane perpendicular to the direction of propagation and the electric field strengths at P, Q and R are identical. If you gave a different answer, you may have misinterpreted Figure 7.10. This indicates the z-dependence of the electric and magnetic fields but does not (and cannot) indicate dependence of the fields on x or y. The lack of x- or y-dependence follows from the plane-wave nature of the wave.

Solution 7.4 (a) The time-varying electric field in an electromagnetic wave drives a current to and fro along the length of a straight aerial. The aerial is most effective when it is oriented parallel to the line of oscillation of the electric field.

(b) The time-varying magnetic field in an electromagnetic wave induces an emf around a circular-loop aerial. The aerial is most effective when the plane of the loop is perpendicular to the line of oscillation of the magnetic field. The plane of the loop then contains the line of oscillation of the electric field and the direction of propagation of the wave.

Solutions for Book 2

Book 2 Chapter 1

Solution 1.1 (a) There is spherical symmetry (and so we will use spherical coordinates).

(b) Using the spherical symmetry, we can conclude that

$$\mathbf{E}(\mathbf{r}) = E_r(r)\mathbf{e}_r.$$

First consider the case $r \le R$. We will use the integral form of Gauss's law,

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, \mathrm{d}V,$$

with a spherical Gaussian surface of radius r. For this surface $\mathbf{E}$ is everywhere parallel to $\mathrm{d}\mathbf{S}$ and has constant magnitude $E_r(r)$. Thus

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi r^2 E_r(r).$$

The total charge enclosed by the Gaussian surface is

$$\begin{aligned}\int_V \rho \, \mathrm{d}V &= \int_0^r ks \, 4\pi s^2 \, \mathrm{d}s = 4\pi k \int_0^r s^3 \, \mathrm{d}s \\ &= 4\pi k \left[\tfrac{1}{4}\, s^4\right]_{s=0}^{s=r} = \pi k r^4.\end{aligned}$$

So, by Gauss's law,

$$4\pi r^2 E_r(r) = \frac{1}{\varepsilon_0} \pi k r^4,$$

i.e.

$$E_r(r) = \frac{k}{4\varepsilon_0}\, r^2, \quad (r \le R).$$

We now consider the case $r > R$. As above, for a spherical Gaussian surface,

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi r^2 E_r(r).$$

The total charge enclosed by the Gaussian surface is

$$\begin{aligned}\int_V \rho \, \mathrm{d}V &= \int_0^r \rho(s) \, 4\pi s^2 \, \mathrm{d}s \\ &= \int_0^R \rho(s) \, 4\pi s^2 \, \mathrm{d}s,\end{aligned}$$

since $\rho(r) = 0$ for $r > R$. So

$$\int_V \rho \, \mathrm{d}V = 4\pi k \int_0^R s^3 \, \mathrm{d}s = \pi k R^4.$$

Hence, by Gauss's law,

$$4\pi r^2 E_r(r) = \frac{1}{\varepsilon_0}\, \pi k R^4,$$

i.e.

$$E_r(r) = \frac{k}{4\varepsilon_0} \frac{R^4}{r^2} \quad (r > R).$$

Combining the two cases,

$$\mathbf{E} = E_r(r)\mathbf{e}_r = \begin{cases} \dfrac{k}{4\varepsilon_0}\, r^2 \mathbf{e}_r & (r \le R), \\[2ex] \dfrac{k}{4\varepsilon_0} \dfrac{R^4}{r^2} \mathbf{e}_r & (r > R). \end{cases}$$

(c) As $\mathbf{E}(\mathbf{r}) = E_r(r)\mathbf{e}_r$,

$$\operatorname{div} \mathbf{E} = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r}(r^2 E_r).$$

So for $r \le R$,

$$\begin{aligned}\operatorname{div} \mathbf{E} &= \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{k}{4\varepsilon_0} r^4\right) \\ &= \frac{k}{\varepsilon_0} r = \frac{1}{\varepsilon_0}\, \rho(r).\end{aligned}$$

Similarly, for $r > R$,

$$\begin{aligned}\operatorname{div} \mathbf{E} &= \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{kR^4}{4\varepsilon_0}\right) \\ &= 0 = \frac{1}{\varepsilon_0}\, \rho(r).\end{aligned}$$

Hence, the differential form of Gauss's law is satisfied both inside and outside the charge distribution.

Solution 1.2 If the vector field $\mathbf{B}(\mathbf{r})$ represents a magnetic field then it must satisfy the no-monopole law,

$$\operatorname{div} \mathbf{B} = 0.$$

For the vector field $\mathbf{F}$,

$$\begin{aligned}\operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2x - y) + \frac{\partial}{\partial z}(0) \\ &= 1 - 1 + 0 = 0.\end{aligned}$$

Hence $\mathbf{F}$ can represent a magnetic field.

Using Ampère's law, the associated current density is

$$\begin{aligned}\mathbf{J} &= \frac{1}{\mu_0} \operatorname{curl} \mathbf{B} = \frac{1}{\mu_0} \operatorname{curl} \mathbf{F} \\ &= \frac{1}{\mu_0} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 2x - y & 0 \end{vmatrix} \\ &= \frac{2}{\mu_0} \mathbf{e}_z.\end{aligned}$$

For the vector field $\mathbf{G}$,

$$\begin{aligned}\operatorname{div} \mathbf{G} &= \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(4y + 5) + \frac{\partial}{\partial z}(0) \\ &= -2 + 4 + 0 = 2.\end{aligned}$$

Hence $\mathbf{G}$ cannot represent a magnetic field because $\operatorname{div}\mathbf{G} \neq 0$.

Solution 1.3

$$\operatorname{curl}\mathbf{E} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 \sin\left[\omega\left(t - \dfrac{y}{c}\right)\right] & 0 & 0 \end{vmatrix}$$
$$= \frac{E_0\omega}{c}\cos\left[\omega\left(t - \frac{y}{c}\right)\right]\mathbf{e}_z.$$

Using the differential form of Faraday's law,

$$\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{E}$$
$$= -\frac{E_0\omega}{c}\cos\left[\omega\left(t - \frac{y}{c}\right)\right]\mathbf{e}_z.$$

Integrating with respect to t, we obtain

$$\mathbf{B}(\mathbf{r}, t) = -\frac{E_0}{c}\sin\left[\omega\left(t - \frac{y}{c}\right)\right]\mathbf{e}_z + \mathbf{B}_0(\mathbf{r}),$$

where $\mathbf{B}_0(\mathbf{r})$ is an arbitrary field which is independent of time. However this field must satisfy the no-monopole law, so $\operatorname{div}\mathbf{B}_0 = 0$.

Solution 1.4 From Maxwell's equations, in empty space, with no charges and currents,

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\operatorname{curl}\mathbf{B} = \varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}.$$

Taking the curl of the first equation and then using the second equation, we have

$$\operatorname{curl}(\operatorname{curl}\mathbf{E}) = -\frac{\partial}{\partial t}(\operatorname{curl}\mathbf{B})$$
$$= -\frac{\partial}{\partial t}(\varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}) = -\varepsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

But

$$\operatorname{curl}(\operatorname{curl}\mathbf{E}) = \operatorname{grad}(\operatorname{div}\mathbf{E}) - \nabla^2\mathbf{E}.$$

Also in empty space, by Gauss's law,

$$\operatorname{div}\mathbf{E} = 0.$$

Hence

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2},$$

as required, where

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}.$$

Solution 1.5 The force on the electron is given by the Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

where $q = -e$. Now

$$\mathbf{E} = E\mathbf{e}_x$$

and

$$\mathbf{v} \times \mathbf{B} = (v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z) \times (B\mathbf{e}_z)$$
$$= v_yB\mathbf{e}_x - v_xB\mathbf{e}_y.$$

So the force acting on the electron is

$$\mathbf{F} = -(eE + eBv_y)\mathbf{e}_x + eBv_x\mathbf{e}_y$$

and the equations of motion are

$$m\dot{v}_x = -(eE + eBv_y),$$
$$m\dot{v}_y = eBv_x$$
$$\text{and} \quad m\dot{v}_z = 0.$$

Comment: The motion is a combination of a circular motion with angular frequency $\omega = eB/m$ in the xy-plane, a uniform motion in the negative y-direction with speed E/B and a uniform motion in the z-direction. This type of trajectory was used in the original determination of e/m for photoelectrons.

Book 2 Chapter 2

Solution 2.1

(a) The total number of atoms in the sphere is

$$N = (2.17 \times 10^{28}\,\mathrm{m^{-3}})$$
$$\times\, (\tfrac{4}{3}\pi \times (1.50 \times 10^{-3})^3\,\mathrm{m^3})$$
$$= 3.07 \times 10^{20}.$$

So the magnitude of the average dipole moment of an argon atom is

$$\langle p\rangle = \frac{\Pi}{N}$$
$$= \frac{5.50 \times 10^{-14}\,\mathrm{C\,m}}{3.07 \times 10^{20}}$$
$$= 1.79 \times 10^{-34}\,\mathrm{C\,m}.$$

The direction of $\langle \mathbf{p}\rangle$ is the same as that of $\mathbf{\Pi}$.

The separation of the centres of charge in this dipole is

$$d = \frac{\langle p\rangle}{q} = \frac{\langle p\rangle}{Ze}$$
$$= \frac{1.79 \times 10^{-34}\,\mathrm{C\,m}}{18 \times (1.60 \times 10^{-19}\,\mathrm{C})}$$
$$= 6.23 \times 10^{-17}\,\mathrm{m}.$$

(You might like to compare this with the diameter of the argon atom, which is $3.4 \times 10^{-10}\,\mathrm{m}$. The charge separation is very small compared to the atomic diameter.)

(b) The polarization $\mathbf{P}$ of the sphere is in the same direction as $\langle \mathbf{p}\rangle$ and has magnitude

$$P = n\langle p\rangle$$
$$= (2.17 \times 10^{28}\,\mathrm{m^{-3}}) \times (1.79 \times 10^{-34}\,\mathrm{C\,m})$$
$$= 3.89 \times 10^{-6}\,\mathrm{C\,m^{-2}}.$$

Solution 2.2 (a) The bound volume charge density is

$$\rho_{\rm b} = -\operatorname{div}\mathbf{P}.$$

Using spherical coordinates

$$\begin{aligned}\operatorname{div}\mathbf{P} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, P_\theta) \\ &\quad + \frac{1}{r\sin\theta}\frac{\partial P_\phi}{\partial\phi} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(ar^3\cos^2\theta) \\ &\quad - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(ar\cos\theta\sin^2\theta) \\ &= 3a\cos^2\theta - \frac{a}{\sin\theta}(-\sin^3\theta + 2\cos^2\theta\sin\theta) \\ &= 3a\cos^2\theta + a\sin^2\theta - 2a\cos^2\theta \\ &= a.\end{aligned}$$

So $\rho_{\rm b} = -a$.

(b) The bound surface charge density is

$$\sigma_{\rm b} = \mathbf{P}\cdot\widehat{\mathbf{n}}.$$

The outward-pointing unit normal to the surface of the sphere is $\widehat{\mathbf{n}} = \mathbf{e}_r$, and on the surface of the sphere $r = R$. So

$$\begin{aligned}\sigma_{\rm b} &= (aR\cos^2\theta\,\mathbf{e}_r - aR\cos\theta\,\sin\theta\,\mathbf{e}_\theta)\cdot\mathbf{e}_r \\ &= aR\cos^2\theta.\end{aligned}$$

Solution 2.3 By symmetry, the electric field **E** and electric displacement **D** inside the dielectric material are in the same direction as the external electric field. Also in an LIH material, $\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$.

Furthermore, at the boundary between the dielectric material and air, $D_\perp$ is continuous. As the boundary is perpendicular to the direction of the electric displacement, the magnitude of the electric displacement in the dielectric slab has magnitude

$$\begin{aligned}D_{\rm slab} &= D_{\rm air} \\ &= \varepsilon_0 E_{\rm air} \\ &= (8.85\times10^{-12}\times 12\,000)\,\mathrm{C\,m^{-2}} \\ &= 1.1\times10^{-7}\,\mathrm{C\,m^{-2}}.\end{aligned}$$

So the electric field in the slab has magnitude

$$\begin{aligned}E_{\rm slab} &= D_{\rm slab}/\varepsilon\varepsilon_0 \\ &= E_{\rm air}/\varepsilon \\ &= 12\,000/6\,\mathrm{V\,m^{-1}} \\ &= 2.0\times10^{3}\,\mathrm{V\,m^{-1}}.\end{aligned}$$

The polarization in the slab has magnitude

$$\begin{aligned}P_{\rm slab} &= \chi_E\varepsilon_0 E_{\rm slab} \\ &= (\varepsilon-1)\varepsilon_0 E_{\rm slab} \\ &= (5.0\times 8.85\times10^{-12}\times 2000)\,\mathrm{C\,m^{-2}} \\ &= 8.9\times10^{-8}\,\mathrm{C\,m^{-2}}.\end{aligned}$$

Solution 2.4 (a) Neglecting edge effects, the field lines of **D** (and **E**) will be perpendicular to the plates.

We apply Gauss's law

$$\int_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho_{\rm f}\,\mathrm{d}V$$

to a pillbox with one end in plate 1 and the other end in material 1.

Within the conducting plate, **D** is zero, and so Gauss's law leads to

$$D_1 = \sigma$$

at all positions in material 1. By the continuity of $D_\perp$ at the interface between the dielectrics,

$$D_2 = D_1 = \sigma.$$

Using the integral form of Gauss's law for a similar pillbox with one end in plate 2, we find that the charge density at plate 2 is $-\sigma$. (Alternatively we could have used a pillbox with one end in plate 1 and the other end in plate 2 and concluded that the total charge in this pillbox is zero.)

Now

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}.$$

So

$$E_1 = \frac{1}{\varepsilon_1\varepsilon_0}D_1 = \frac{1}{\varepsilon_1\varepsilon_0}\sigma$$

and $$E_2 = \frac{1}{\varepsilon_2\varepsilon_0}D_2 = \frac{1}{\varepsilon_2\varepsilon_0}\sigma.$$

Also the polarization

$$\mathbf{P} = \chi_E\varepsilon_0\mathbf{E} = (\varepsilon-1)\varepsilon_0\mathbf{E}.$$

So

$$P_1 = \frac{\varepsilon_1-1}{\varepsilon_1}\sigma \quad\text{and}\quad P_2 = \frac{\varepsilon_2-1}{\varepsilon_2}\sigma.$$

(b) The potential difference across dielectric 1 is

$$d_1E_1 = \frac{d_1}{\varepsilon_1\varepsilon_0}\sigma$$

and across dielectric 2 is

$$d_2E_2 = \frac{d_2}{\varepsilon_2\varepsilon_0}\sigma.$$

So the total potential difference across the capacitor is

$$V = \frac{1}{\varepsilon_0}\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right)\sigma.$$

Now the capacitance is

$$C = \frac{Q}{V} = \frac{\sigma A}{V},$$

so that

$$C = \frac{\varepsilon_0 A}{\left(\dfrac{d_1}{\varepsilon_1} + \dfrac{d_2}{\varepsilon_2}\right)}.$$

Book 2 Chapter 3

Solution 3.1 We will use cylindrical coordinates, with the z-axis along the axis of the wire and in the direction of the electric current. The magnetic intensity $\mathbf{H}$ and the magnetic field $\mathbf{B}$ are azimuthal, i.e. in the direction of $\mathbf{e}_\phi$.

Using Ampère's law,

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \mathbf{J}_\mathrm{f} \cdot \mathrm{d}\mathbf{S},$$

for a circular path of radius r centred on the axis of symmetry of the wire,

$$2\pi r H = \pi r^2 J,$$

which gives

$$H = \tfrac{1}{2} r J.$$

Now $\mathbf{B} = \mu\mu_0\mathbf{H}$ and so

$$B = \tfrac{1}{2}\mu\mu_0 r J.$$

Solution 3.2 In the absence of free surface currents, the components $B_\perp$ and $H_\|$ are continuous.

For LIH materials, $\mathbf{H} = \mathbf{B}/\mu\mu_0$, so $\mathbf{H}$ is in the same direction as $\mathbf{B}$.

The continuity of $B_\perp$ at the interface gives

$$B_1 \cos 60^\circ = B_2 \cos\theta,$$

whereas the continuity of $H_\|$ gives

$$\frac{1}{\mu_1\mu_0} B_1 \sin 60^\circ = \frac{1}{\mu_2\mu_0} B_2 \sin\theta.$$

Dividing these last two equations leads to

$$\frac{1}{\mu_1}\tan 60^\circ = \frac{1}{\mu_2}\tan\theta.$$

So

$$\tan\theta = \frac{\mu_2}{\mu_1}\tan 60^\circ = \frac{3.0}{1.5}\sqrt{3}.$$

Hence

$$\theta = \tan^{-1}(2\sqrt{3}) = 74^\circ.$$

Solution 3.3 (a) The magnetic field lines are circular lines around the toroidal ring.

We apply Ampère's law,

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \mathbf{J}_\mathrm{f} \cdot \mathrm{d}\mathbf{S},$$

to a circular path C of radius r within the ring (as shown in Figure 3.20 in Book 2).

If there are N turns carrying a current I, the total current flowing through the disc S is NI. So Ampère's law gives

$$2\pi r H = NI.$$

Hence

$$\begin{aligned} H &= \frac{NI}{2\pi r} \\ &= \frac{2000 \times 1.5}{2\pi \times 0.12}\,\mathrm{A\,m^{-1}} \\ &= 3980\,\mathrm{A\,m^{-1}} = 4.0 \times 10^3\,\mathrm{A\,m^{-1}} \text{ to 2 sig. figs.} \end{aligned}$$

The magnitude of the magnetic field is

$$\begin{aligned} B &= \mu\mu_0 H \\ &= (400 \times 4\pi \times 10^{-7} \times 3980)\,\mathrm{T} \\ &= 2.0\,\mathrm{T}. \end{aligned}$$

(b) At the interface between the air in the gap and the iron alloy, the perpendicular component of the magnetic field $\mathbf{B}$ is continuous. So

$$B_\mathrm{air} = B_\mathrm{iron}.$$

It follows that

$$H_\mathrm{air} = \mu H_\mathrm{iron}.$$

We again use Ampère's law for the same circular path C as used in part (a). In this case we obtain

$$(2\pi r - d)H_\mathrm{iron} + dH_\mathrm{air} = NI,$$

where d is the width of the air-gap.

So

$$\begin{aligned} H_\mathrm{iron} &= \frac{NI}{(2\pi r - d) + \mu d} \\ &= \frac{NI}{2\pi r + (\mu - 1)d} \\ &= \frac{2000 \times 1.5}{2\pi \times 0.12 + 399 \times 0.003}\,\mathrm{A\,m^{-1}} \\ &= 1538\,\mathrm{A\,m^{-1}} = 1.5 \times 10^3\,\mathrm{A\,m^{-1}} \\ &\qquad \text{to 2 sig. figs.} \end{aligned}$$

Now

$$\begin{aligned} H_\mathrm{air} &= \mu H_\mathrm{iron} \\ &= 400 \times 1538\,\mathrm{A\,m^{-1}} \\ &= 6.152 \times 10^5\,\mathrm{A\,m^{-1}} = 6.2 \times 10^5\,\mathrm{A\,m^{-1}} \\ &\qquad \text{to 2 sig. figs.} \end{aligned}$$

Using $\mathbf{B} = \mu\mu_0\mathbf{H}$ for either the iron ring or the air in the gap, we obtain

$$B_\mathrm{iron} = B_\mathrm{air} = 0.773\,\mathrm{T} = 0.77\,\mathrm{T} \text{ to 2 sig. figs.}$$

Solution 3.4 The bound current density is given by

$$\mathbf{J}_\mathrm{b} = \operatorname{curl}\mathbf{M},$$

where $\mathbf{M}(r)$ is the magnetization. In a linear, isotropic, homogeneous material

$$\mathbf{M} = \frac{\chi_B}{\mu_0}\mathbf{B} = \mu\chi_B\mathbf{H},$$

where μ and χ_B are constants. Hence

$$\mathbf{J}_\mathrm{b} = \mu\chi_B \operatorname{curl}\mathbf{H}.$$

But, from Ampère's law

$$\operatorname{curl}\mathbf{H} = \mathbf{J}_\mathrm{f},$$

and so

$$\mathbf{J}_\mathrm{b} = \mu\chi_B\mathbf{J}_\mathrm{f}.$$

In other words, the bound and free current densities are proportional to one another. In particular when the material carries no free current, the magnetization current is zero at all points in the volume of the material, leaving only the possibility of bound surface currents $\mathbf{i}_\mathrm{b} = \mathbf{M}\times\widehat{\mathbf{n}}$.

Comments: Inhomogeneities may exist at, or near, the surface of a material, so that bound current density may exist in the surface layer. Furthermore, non-linear materials (such as ferromagnets) can certainly have non-zero bound current densities in the absence of free currents.

Book 2 Chapter 4

Solution 4.1 (a) This form of the differential version of Gauss's law would be used in regions where the total charge density ρ is known.

(b) This form of the differential version of Gauss's law would be used in regions which contain dielectric materials, and where the free charge density ρ_f is known.

(c) Gauss's law in its integral version would be used in regions where the total charge density ρ is known, and where there is a high degree of symmetry.

(d) This form of the integral version of Gauss's law would be used in regions which contain dielectric materials, where the free charge density ρ_f is known, and where there is a high degree of symmetry.

(e) This version of Poisson's equation would be used in regions where the total charge density ρ is known, as in part (a).

(f) This version of Poisson's equation would be used in regions which contain an LIH dielectric material and where the free charge density is known.

(g) Laplace's equation would be used when the total charge density $\rho(\mathbf{r}) = 0$ everywhere in the region.

Solution 4.2 The potential $V(x)$ in the dielectric material satisfies Poisson's equation

$$\nabla^2 V = -\frac{1}{\varepsilon\varepsilon_0}\,\rho_\mathrm{f}$$

and because of the symmetry this reduces to

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -\frac{1}{\varepsilon\varepsilon_0}\,(A + Bx) \qquad (0 < x < s)\,.$$

Integrating, we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{1}{\varepsilon\varepsilon_0}\,\left(Ax + \tfrac{1}{2}Bx^2\right) + C$$

and

$$V(x) = -\frac{1}{\varepsilon\varepsilon_0}\left(\tfrac{1}{2}Ax^2 + \tfrac{1}{6}Bx^3\right) + Cx + D,$$

where C and D are constants of integration.

We apply the boundary conditions at $x = 0$ and $x = s$ to determine C and D.

At $x = 0$, $V(0) = 0$ and so $D = 0$.

At $x = s$, $V(s) = V_s$ and so

$$V_s = -\frac{1}{\varepsilon\varepsilon_0}\left(\tfrac{1}{2}As^2 + \tfrac{1}{6}Bs^3\right) + Cs.$$

Hence

$$C = \frac{V_s}{s} + \frac{1}{\varepsilon\varepsilon_0}\left(\tfrac{1}{2}As + \tfrac{1}{6}Bs^2\right).$$

Substituting for C and D in the expression for the potential,

$$\begin{aligned}V(x) &= -\frac{1}{\varepsilon\varepsilon_0}\left(\tfrac{1}{2}Ax^2 + \tfrac{1}{6}Bx^3\right) + V_s\,\frac{x}{s}\\ &\quad + \frac{x}{\varepsilon\varepsilon_0}\left(\tfrac{1}{2}As + \tfrac{1}{6}Bs^2\right)\\ &= V_s\frac{x}{s} - \frac{1}{\varepsilon\varepsilon_0}\Big[\tfrac{1}{2}A(x^2 - sx)\\ &\quad + \tfrac{1}{6}B(x^3 - s^2x)\Big].\end{aligned}$$

The electric field is given by

$$\begin{aligned}\mathbf{E} &= -\operatorname{grad}V = -\frac{\mathrm{d}V}{\mathrm{d}x}\,\mathbf{e}_x\\ &= -\Bigg(\frac{V_s}{s} - \frac{1}{\varepsilon\varepsilon_0}\Big[\tfrac{1}{2}A\,(2x - s)\\ &\quad + \tfrac{1}{6}B\left(3x^2 - s^2\right)\Big]\Bigg)\,\mathbf{e}_x.\end{aligned}$$

Solution 4.3 (a) The electrostatic potential V in the dielectric material satisfies Poisson's equation

$$\nabla^2 V = -\frac{1}{\varepsilon\varepsilon_0}\rho_\mathrm{f}.$$

The system has spherical symmetry, and $\rho_\mathrm{f} = \rho_0$, so this reduces to

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}V}{\mathrm{d}r}\right) = -\frac{1}{\varepsilon\varepsilon_0}\,\rho_0 \qquad (a < r < 2a).$$

(b) Integrating, we obtain

$$r^2\,\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{\rho_0}{3\varepsilon\varepsilon_0}\,r^3 + A$$

and

$$V(r) = -\frac{\rho_0}{6\varepsilon\varepsilon_0}\,r^2 - \frac{A}{r} + B,$$

where A and B are constants of integration.

(c) In order to find the constants A and B we use the boundary conditions $V(a) = 0$ and $V(2a) = 0$. So

$$V(a) = 0 = -\frac{\rho_0}{6\varepsilon\varepsilon_0}\,a^2 - \frac{A}{a} + B,$$

and

$$V(2a) = 0 = -\frac{2\rho_0}{3\varepsilon\varepsilon_0}\,a^2 - \frac{A}{2a} + B.$$

$$\text{Hence}\quad A = \frac{\rho_0 a^3}{\varepsilon\varepsilon_0} \quad \text{and} \quad B = \frac{7\rho_0 a^2}{6\varepsilon\varepsilon_0}.$$

Therefore

$$\begin{aligned} V(r) &= -\frac{\rho_0}{6\varepsilon\varepsilon_0}\,r^2 - \frac{\rho_0 a^3}{\varepsilon\varepsilon_0}\frac{1}{r} + \frac{7\rho_0 a^2}{6\varepsilon\varepsilon_0} \\ &= -\frac{\rho_0 r^2}{6\varepsilon\varepsilon_0}\left[1 + 6\left(\frac{a}{r}\right)^3 - 7\left(\frac{a}{r}\right)^2\right] \\ &\qquad (a \leq r \leq 2a)\,. \end{aligned}$$

Solution 4.4 The conducting plate can be 'replaced' by two image charges, each $-q$, located at points $(-d/2,\, d/2,\, 0)$ and $(-d/2,\, -d/2,\, 0)$ (see Figure 12).

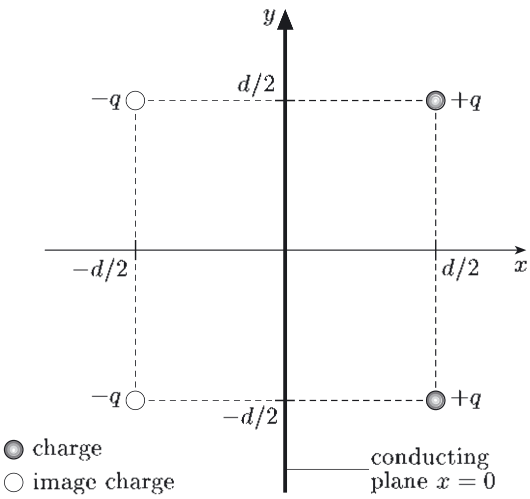

Figure 12 For Solution 4.4.

By symmetry, the real charges and the image charges produce zero electrostatic potential in the plane $x = 0$ midway between them, and the potential very close to each of the real charges will be unaffected by the image charges. So by the uniqueness theorem, since the boundary conditions are the same, the solution for the electrostatic fields and potentials in the region $x > 0$ for the problem with real and image charges must be the same as the solution for the problem with the real charges and the conducting plate. And since the force on charge q at $(d/2,\, d/2,\, 0)$ is just the charge times the electric field at that point, we can determine the force by adding the Coulomb forces due to the other real charge and due to the two image charges.

The electrostatic force on q at $(d/2,\, d/2,\, 0)$ is therefore (see Figure 13)

$$\begin{aligned} \mathbf{F} = \mathbf{F}&\big(\text{due to charge} + q \text{ at } (d/2,\, -d/2,\, 0)\big) \\ &+ \mathbf{F}\big(\text{due to charge} - q \text{ at } (-d/2,\, +d/2,\, 0)\big) \\ &+ \mathbf{F}\big(\text{due to charge} - q \text{ at } (-d/2,\, -d/2,\, 0)\big). \end{aligned}$$

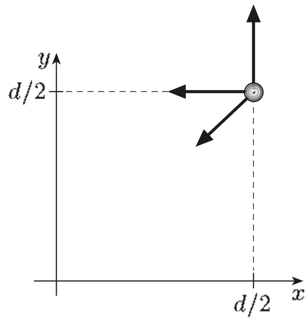

Figure 13 For Solution 4.4.

These three contributions to the force are in the directions of the unit vectors $\mathbf{e}_y$, $-\mathbf{e}_x$ and $-(\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$, respectively, and the separations of the charges are d, d and $\sqrt{2}d$, respectively. So the force is

$$\begin{aligned} \mathbf{F} &= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{d^2}\,\mathbf{e}_y - \frac{1}{d^2}\,\mathbf{e}_x - \frac{1}{2d^2}\frac{(\mathbf{e}_x + \mathbf{e}_y)}{\sqrt{2}}\right] \\ &= \frac{q^2}{4\pi\varepsilon_0 d^2}\left[-\left(1 + \frac{1}{2\sqrt{2}}\right)\mathbf{e}_x + \left(1 - \frac{1}{2\sqrt{2}}\right)\mathbf{e}_y\right]. \end{aligned}$$

The magnitude of this force is

$$\begin{aligned} F &= \frac{q^2}{4\pi\varepsilon_0 d^2}\left[\left(1 + \frac{1}{2\sqrt{2}}\right)^2 + \left(1 - \frac{1}{2\sqrt{2}}\right)^2\right]^{1/2} \\ &= \frac{q^2}{4\pi\varepsilon_0 d^2}\left[\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{8}\right) + \left(1 - \frac{1}{\sqrt{2}} + \frac{1}{8}\right)\right]^{1/2} \\ &= \frac{3q^2}{8\pi\varepsilon_0 d^2}. \end{aligned}$$

Solution 4.5 In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

For $V(r,\, \phi,\, z) = Ar\cos\phi + (B\cos\phi)/r$,

$$\begin{aligned} \frac{\partial V}{\partial r} &= A\cos\phi - \frac{B\cos\phi}{r^2}, \\ \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) &= \frac{\partial}{\partial r}\left(Ar\cos\phi - \frac{B\cos\phi}{r}\right) \\ &= A\cos\phi + \frac{B\cos\phi}{r^2}, \end{aligned}$$

$$\begin{aligned} \frac{\partial V}{\partial\phi} &= -Ar\sin\phi - \frac{B\sin\phi}{r}, \\ \frac{\partial^2 V}{\partial\phi^2} &= -Ar\cos\phi - \frac{B\cos\phi}{r}, \\ \frac{\partial^2 V}{\partial z^2} &= 0. \end{aligned}$$

So

$$\begin{aligned} \nabla^2 V &= \frac{1}{r}\left(A\cos\phi + \frac{B\cos\phi}{r^2}\right) \\ &\quad + \frac{1}{r^2}\left(-Ar\cos\phi - \frac{B\cos\phi}{r}\right) + 0 = 0, \end{aligned}$$

as required.

(b) We use cylindrical coordinates with the z-axis along the axis of the cylinder and the line $\phi = 0$ in the direction of the uniform electric field $\mathbf{E}_0$. The potential corresponding to this uniform electric field is then $V = -E_0 r \cos\phi$.

We shall denote the potential outside the cylinder by $V_1(\mathbf{r})$ and inside the cylinder by $V_2(\mathbf{r})$. These potentials must satisfy Laplace's equation with the following boundary conditions:

(i) $V_1 \simeq -E_0 r \cos\phi$ for large values of r;

(ii) $V_1 = V_2$ for $r = R$;

(iii) $\partial V_1/\partial r = \varepsilon\, \partial V_2/\partial r$ for $r = R$;

(iv) V_2 remains finite for $r = 0$.

Part (a) and boundary conditions (i) and (iv) suggest that we look for solutions of the form

$$V_1 = -E_0 r \cos\phi + \frac{B\cos\phi}{r} \qquad \text{for } r \geq R,$$
$$V_2 = Ar\cos\phi \qquad \text{for } r < R.$$

The coefficient of $r\cos\phi$ in V_1 ensures that boundary condition (i) is satisfied, whereas the absence of a term proportional to $(1/r)\ \cos\phi$ in V_2 ensures that boundary condition (iv) is satisfied. It remains to choose values for the constants A and B so that boundary conditions (ii) and (iii) are satisfied. Using boundary condition (ii), we obtain

$$-E_0 R + \frac{B}{R} = AR,$$

whereas boundary condition (iii) leads to

$$-E_0 - \frac{B}{R^2} = \varepsilon A.$$

Solving these two equations,

$$A = -\frac{2}{\varepsilon + 1} E_0, \quad \text{and} \quad B = \frac{\varepsilon - 1}{\varepsilon + 1} E_0 R^2.$$

So the electrostatic potential is

$$V_1 = -E_0 r \cos\phi + \frac{\varepsilon - 1}{\varepsilon + 1}\, E_0 R^2\, \frac{\cos\phi}{r} \qquad \text{for } r \geq R,$$
$$V_2 = -\frac{2}{\varepsilon + 1}\, E_0 r \cos\phi \qquad \text{for } r < R.$$

Note that the electric field inside the cylinder is uniform, but reduced by a factor $2/(\varepsilon + 1)$ compared to the applied electric field.

Solution 4.6 Amending the argument used to derive Equation 4.26 of Book 2, the general finite difference form of Laplace's equation in *three* dimensions is

$$\begin{aligned}
\nabla^2 V = {} & \frac{V(x_{i+1}, y_j, z_k) - 2V(x_i, y_j, z_k) + V(x_{i-1}, y_j, z_k)}{(\Delta x)^2} \\
& + \frac{V(x_i, y_{j+1}, z_k) - 2V(x_i, y_j, z_k) + V(x_i, y_{j-1}, z_k)}{(\Delta y)^2} \\
& + \frac{V(x_i, y_j, z_{k+1}) - 2V(x_i, y_j, z_k) + V(x_i, y_j, z_{k-1})}{(\Delta z)^2} \\
= {} & 0.
\end{aligned}$$

If we take the step lengths Δx, Δy and Δz to be equal, this reduces to

$$\begin{aligned}
V(x_i, y_j, z_k) = \frac{1}{6}\Big[& V(x_{i+1}, y_j, z_k) \\
& + V(x_i, y_{j+1}, z_k) + V(x_i, y_j, z_{k+1}) \\
& + V(x_{i-1}, y_j, z_k) + V(x_i, y_{j-1}, z_k) \\
& + V(x_i, y_j, z_{k-1})\Big].
\end{aligned}$$

This equation says that the value of V at any grid point is the average of the values of V at the six closest neighbouring grid points.

Solution 4.7 We use Equation 4.27 of Book 2 iteratively to improve on the initial guesses for the potentials at the nine internal grid points. For the four corner points,

$$V = \tfrac{1}{4}(0 + 1 + 0.5 + 0.5) = 0.5.$$

For the midpoints at the top and bottom,

$$V = \tfrac{1}{4}(0 + 0.5 + 0.5 + 0.5) = 0.375.$$

For the midpoints on left and right,

$$V = \tfrac{1}{4}(1 + 0.5 + 0.5 + 0.5) = 0.625.$$

For the central point,

$$V = \tfrac{1}{4}(0.5 + 0.5 + 0.5 + 0.5) = 0.5.$$

These results are shown below.

	0	**0**	**0**	
1	0.5	0.375	0.5	**1**
1	0.625	0.5	0.625	**1**
1	0.5	0.375	0.5	**1**
	0	**0**	**0**	

A second iteration leads to the following values for V. For the four corner points,

$$V = \tfrac{1}{4}(0 + 1 + 0.625 + 0.375) = 0.5.$$

For the midpoints at the top and bottom,

$$V = \tfrac{1}{4}(0 + 0.5 + 0.5 + 0.5) = 0.375.$$

For the midpoints on left and right,

$$V = \tfrac{1}{4}(1 + 0.5 + 0.5 + 0.5) = 0.625.$$

For the central point,

$$V = \tfrac{1}{4}(0.375 + 0.625 + 0.375 + 0.625) = 0.5.$$

The values after the second iteration are identical to those after a single iteration. So the Jacobi relaxation

method has converged and no further iterations are necessary.

Book 2 Chapter 5

Solution 5.1 The field due to a current element is given by the Biot–Savart law:

$$\delta\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\frac{I\,\delta\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$

In this case,

$$\begin{aligned} I\,\delta\mathbf{l}' &= 5.0\times 10^{-6}\ \mathbf{e}_y\ \mathrm{A\,m},\\ \mathbf{r} &= (0.08\,\mathbf{e}_x + 0.08\,\mathbf{e}_y)\,\mathrm{m},\\ \mathbf{r}' &= 0.02\,\mathbf{e}_x\ \mathrm{m},\\ \mathbf{r}-\mathbf{r}' &= (0.06\,\mathbf{e}_x + 0.08\,\mathbf{e}_y)\,\mathrm{m},\\ |\mathbf{r}-\mathbf{r}'| &= \sqrt{0.06^2+0.08^2}\ \mathrm{m} = 0.1\,\mathrm{m}. \end{aligned}$$

Also,

$$\begin{aligned} &I\,\delta\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')\\ &= \left(5.0\times 10^{-6}\ \mathbf{e}_y \times (0.06\,\mathbf{e}_x + 0.08\,\mathbf{e}_y)\right)\ \mathrm{A\,m}^2\\ &= 0.3\times 10^{-6}\,(\mathbf{e}_y\times\mathbf{e}_x)\ \mathrm{A\,m}^2\\ &= -\,0.3\times 10^{-6}\,\mathbf{e}_z\ \mathrm{A\,m}^2. \end{aligned}$$

Substituting these results into the Biot–Savart law,

$$\begin{aligned} \delta\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\,\frac{I\,\delta\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\\ &= 10^{-7}\,\mathrm{N\,A}^{-2}\,\frac{(-0.3\times 10^{-6}\,\mathbf{e}_z\ \mathrm{A\,m}^2)}{(0.1\,\mathrm{m})^3}\\ &= -3\times 10^{-11}\,\mathbf{e}_z\ \mathrm{N\,A}^{-1}\,\mathrm{m}^{-1}\\ &= -3\times 10^{-11}\,\mathbf{e}_z\ \mathrm{T}. \end{aligned}$$

Solution 5.2 By symmetry, each side of the hexagon will produce a field of equal magnitude at the centre, and all of the contributions will be in the same direction, perpendicular to the plane of the loop. The magnitude of the field produced by one side is given by Equation 5.8 in Book 2:

$$B = \frac{\mu_0 I}{4\pi d}\,[\sin\alpha_{\mathrm{B}} - \sin\alpha_{\mathrm{A}}],$$

where the distance d between field point and wire in this case is $L\sin\pi/3 = L\sin 60^\circ$ (the height of an equilateral triangle with side length L), and $\alpha_{\mathrm{B}} = -\alpha_{\mathrm{A}} = \pi/6 = 30^\circ$. So

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi L\sin 60^\circ}\,[2\sin 30^\circ]\\ &= \frac{\mu_0 I}{4\pi L\sqrt{3}/2}\,[2\times\tfrac{1}{2}]\\ &= \frac{\mu_0 I}{2\sqrt{3}\pi L}. \end{aligned}$$

So the total field at the centre of the hexagon has a magnitude

$$B_{\mathrm{hex}} = 6\,\frac{\mu_0 I}{2\sqrt{3}\pi L} = \frac{\sqrt{3}\mu_0 I}{\pi L}.$$

Solution 5.3 (a) The magnitude of the axial field in a solenoid of finite length, with number of turns per unit length n, is given by Equation 5.13 in Book 2:

$$B = \frac{\mu_0 n I}{2}\,[\sin\alpha_{\mathrm{B}} - \sin\alpha_{\mathrm{A}}].$$

In the middle of a long solenoid, for which we assume that the diameter is very much less than the length, $\alpha_{\mathrm{A}} = -\pi/2$ and $\alpha_{\mathrm{B}} = \pi/2$, so

$$B = \mu_0 n I,$$

which is a standard result. At one end of a long solenoid, $\alpha_{\mathrm{A}} = 0$ and $\alpha_{\mathrm{B}} = \pi/2$, so

$$B = \mu_0 n I/2,$$

which is half of the value obtained for the field in the middle.

(b) In the middle of a solenoid whose length is 10 times its diameter,

$$\sin\alpha_{\mathrm{B}} = -\sin\alpha_{\mathrm{A}} = \frac{10}{\sqrt{1^2+10^2}} = \frac{10}{\sqrt{101}}.$$

So Equation 5.13 gives

$$B = \frac{\mu_0 n I}{2}\,\frac{20}{\sqrt{101}} \simeq 0.995\,\mu_0 n I,$$

which is only 0.5% less than the result for a solenoid of infinite length.

Solution 5.4 (a)

$$\begin{aligned} \mathbf{B} = \operatorname{curl}\mathbf{A} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z \end{vmatrix}\\ &= -\tfrac{1}{2}\,a\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & z^2 & x^2 \end{vmatrix}\\ &= a(z\mathbf{e}_x + x\mathbf{e}_y + y\mathbf{e}_z). \end{aligned}$$

(b) Using the differential version of Ampère's law,

$$\begin{aligned} \mathbf{J} &= \frac{1}{\mu_0}\operatorname{curl}\mathbf{B}\\ &= \frac{1}{\mu_0}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ az & ax & ay \end{vmatrix}\\ &= \frac{1}{\mu_0}(a+a+a)\\ &= \frac{3a}{\mu_0}. \end{aligned}$$

Solution 5.5 (a) As discussed in Subsection 5.3.2, for the spherical-head approximation, *radial* dipoles and their return currents give rise to fields outside the scalp that are *tangential* to the surface of the scalp.

A dipole that is perpendicular to the surface is a radial dipole, so in this case there will be no radial component to the magnetic field just outside the scalp. Since the magnetometer only measures the radial component, the measured field will be zero.

(b) A dipole parallel to the surface is a *tangential* dipole, and the radial component of the magnetic field outside the scalp is identical to the radial field due to the dipole alone; there is no contribution to the radial field from the return currents (see discussion in Subsection 5.3.2). The magnetic field produced by the dipole is in the azimuthal direction about the dipole axis, that is, the field lines are circular. Directly above the dipole the field is parallel to the surface of the scalp, and so there is no radial component of the field, so again the magnetometer will not register a field.

(c) A dipole $I\,\delta\mathbf{l}$ at $45°$ to the surface can be resolved into component dipoles of equal magnitude, $I\,\delta l\,\sin 45° = I\,\delta l\,\cos 45°$, in the radial and tangential directions. Neither of these component dipoles produces a radial component of magnetic field at the point directly above it, so the magnetometer does not register a field from the inclined dipole.

Solution 5.6 The measured radial field is that due to the dipole alone, with no contribution from the return current. As discussed in the solution to Exercise 5.5, the field lines are circles, centred on the axis of the dipole. So on one side of the highest point of the head the radial field will have an outward component and on the other side it will have an inward component. A line joining the position of the peak in the radially outward field to the position of the peak in the inwardly pointing field will be perpendicular to the direction of the current dipole, and the separation of the peaks will increase as the depth of the dipole increases. The magnitudes of the peak fields is a measure of the strength of the current dipole. (See Figure 5.25 and accompanying discussion.)

Book 2 Chapter 6

Solution 6.1 Problems like this require a systematic approach to keep track of the different components of the forces. The resultant force is

$$\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Now

$$\mathbf{v} \times \mathbf{B} = 2.0 \times 10^6\,\mathrm{m\,s^{-1}} \times 2.5\,\mathrm{T} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 5.0 \times 10^6\,\mathrm{T\,m\,s^{-1}}(\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z)$$

$$= 5.0 \times 10^6\,\mathrm{V\,m^{-1}}(\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z),$$

since the unit of $\mathbf{v} \times \mathbf{B}$ must be the same as the unit of $\mathbf{E}$. So

$$\mathbf{F} = -1.60 \times 10^{-19}\,\mathrm{C}\,\Big(2.5 \times 10^6\,\mathrm{V\,m^{-1}}\,(\mathbf{e}_x + 2\mathbf{e}_y)$$

$$+5.0 \times 10^6\,\mathrm{V\,m^{-1}}(\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z)\Big)$$

$$= -4.0 \times 10^{-13}\,\mathrm{N}\,(3\mathbf{e}_x + 4\mathbf{e}_y - 2\mathbf{e}_z).$$

The magnitude of the vector $(3\mathbf{e}_x + 4\mathbf{e}_y - 2\mathbf{e}_z)$ is $\sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$, so the magnitude of the force is

$$F = 4.0 \times 10^{-13}\,\mathrm{N} \times \sqrt{29} = 2.2 \times 10^{-12}\,\mathrm{N}.$$

Solution 6.2 At $t = 0$ the particle is at rest so it cannot experience a magnetic force. The initial acceleration is therefore due to an electric force, $\mathbf{F}_{\mathrm{elec}}$. The electric field must be in the same direction as the acceleration of the positively-charged proton, that is, in the $+x$-direction, and its magnitude is

$$E = \frac{F_{\mathrm{elec}}}{e}$$

$$= \frac{ma_0}{e}$$

$$= \frac{1.67 \times 10^{-27}\,\mathrm{kg} \times 9.6 \times 10^{14}\,\mathrm{m\,s^{-2}}}{1.60 \times 10^{-19}\,\mathrm{C}}$$

$$= 1.0 \times 10^7\,\mathrm{V\,m^{-1}}.$$

At time t_1, the electric force is unchanged. The acceleration of the moving proton has the same magnitude as it had initially but the direction has reversed. The magnetic force must therefore be in the opposite direction to the electric force and have twice the magnitude of the electric force, that is, $\mathbf{F}_{\mathrm{mag}} = -2ma_0\mathbf{e}_x$. Since $\mathbf{F}_{\mathrm{mag}} = q\mathbf{v} \times \mathbf{B} = e(-v_1\mathbf{e}_y) \times \mathbf{B}$, we can write

$$-2ma_0\mathbf{e}_x = e(-v_1\mathbf{e}_y) \times \mathbf{B} = -ev_1(-B_x\mathbf{e}_z + B_z\mathbf{e}_x).$$

Equating x-components, we deduce that

$$B_z = \frac{2ma_0}{ev_1}$$

$$= \frac{2 \times 1.67 \times 10^{-27}\,\mathrm{kg} \times 9.6 \times 10^{14}\,\mathrm{m\,s^{-2}}}{1.60 \times 10^{-19}\,\mathrm{C} \times 2.0 \times 10^7\,\mathrm{m\,s^{-1}}}$$

$$= 1.0\,\mathrm{T},$$

and comparing z-components of the force we deduce that $B_x = 0$.

The data in (i) and (ii) do not tell us anything about B_y. However, if there is no motion in the z-direction, then there is no force in this direction, which means that the magnetic field must be in the z-direction, and so $B_y = 0$. If B_y were not equal to zero, then immediately after release, when the particle is moving in the x-direction, there would be a component of magnetic force in the z-direction, which conflicts with the observation that the proton does not move in the z-direction.

Solution 6.3 If a particle, which has charge q and is travelling at velocity $\mathbf{v}$, is not deflected, then the net force acting on it is zero, so the electric and magnetic forces must cancel. Thus

$$q\mathbf{E} = -q\mathbf{v} \times \mathbf{B},$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} = -v\,\mathbf{e}_y \times B\,\mathbf{e}_z = -vB\,\mathbf{e}_x$$
$$= -(3.0 \times 10^7\,\mathrm{m\,s^{-1}} \times 0.50\,\mathrm{T})\,\mathbf{e}_x$$
$$= -1.5 \times 10^7\,\mathrm{V\,m^{-1}}\,\mathbf{e}_x.$$

Note that the expression for $\mathbf{E}$ *does not depend on the mass or charge of the particle, so particles of any mass and any charge that are travelling with velocity* $3.0 \times 10^7\,\mathrm{m\,s^{-1}}\,\mathbf{e}_y$ *will be selected.*

Solution 6.4 The magnetic field does not change the proton's speed, which is $3 \times 10^7\,\mathrm{m\,s^{-1}}$. The period of its motion is

$$T = \frac{2\pi}{\omega_\mathrm{c}} = \frac{2\pi m}{eB}$$
$$= \frac{2\pi \times 1.67 \times 10^{-27}\,\mathrm{kg}}{1.60 \times 10^{-19}\,\mathrm{C} \times 2 \times 10^{-10}\,\mathrm{T}}$$
$$\simeq 300\,\mathrm{s}.$$

The diameter of the orbit is

$$d = 2r_\mathrm{c} = \frac{2mv}{|q|B}$$
$$= \frac{2 \times 1.67 \times 10^{-27}\,\mathrm{kg} \times 3 \times 10^7\,\mathrm{m\,s^{-1}}}{1.60 \times 10^{-19}\,\mathrm{C} \times 2 \times 10^{-10}\,\mathrm{T}}$$
$$\simeq 3 \times 10^9\,\mathrm{m}.$$

This orbit is about 10 *times smaller than the diameter of the Earth's orbit around the Sun, but it is traversed in about 5 minutes. Note that even a weak field can have a substantial effect on a high energy particle. In practice, the cosmic ray protons would follow a helical path about the magnetic field direction, rather than a circular path.*

Solution 6.5 (a) The force on an ion is given by $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Since

$$\mathbf{v} = \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{e}_x + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{e}_y + \frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{e}_z,$$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\mathrm{d}x}{\mathrm{d}t} & \frac{\mathrm{d}y}{\mathrm{d}t} & \frac{\mathrm{d}z}{\mathrm{d}t} \\ 0 & B & 0 \end{vmatrix} = -B\frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{e}_x + B\frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{e}_z.$$

Thus $\quad \mathbf{F} = Q(-B\dfrac{\mathrm{d}z}{\mathrm{d}t}\mathbf{e}_x + E\mathbf{e}_y + B\dfrac{\mathrm{d}x}{\mathrm{d}t}\mathbf{e}_z).$

Applying Newton's second law to the ion, we have

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{F},$$

which in component form gives

$$m\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{e}_x + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{e}_y + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\mathbf{e}_z\right)$$
$$= Q\left(-B\frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{e}_x + E\mathbf{e}_y + B\frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{e}_z\right).$$

Equating components on either side,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{QB}{m}\frac{\mathrm{d}z}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{QE}{m}, \quad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \frac{QB}{m}\frac{\mathrm{d}x}{\mathrm{d}t},$$

as required.

(b) We solve these equations of motion using the initial conditions $\mathbf{r} = \mathbf{0}$ and $\mathrm{d}\mathbf{r}/\mathrm{d}t = v\mathbf{e}_z$ when $t = 0$. Solving the equation for the y-component by integrating twice gives

$$y = \frac{QE}{2m}t^2.$$

Integrating the equation for the x-component of the acceleration, and noting that $\mathrm{d}x/\mathrm{d}t = 0$ when $z = 0$, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{QB}{m}z.$$

Substituting this expression for $\mathrm{d}x/\mathrm{d}t$ into the equation for the z-component of the acceleration gives

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -\left(\frac{QB}{m}\right)^2 z.$$

This is the equation of simple harmonic motion, with the solution

$$z = \frac{mv}{QB}\sin\left(\frac{QB}{m}t\right),$$

which satisfies the boundary conditions $z = 0$ and $\mathrm{d}z/\mathrm{d}t = v$ at $t = 0$. Solving for x,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{QB}{m}z = -v\sin\left(\frac{QB}{m}t\right),$$

and so

$$x = \frac{mv}{QB}\cos\left(\frac{QB}{m}t\right) - \frac{mv}{QB}$$
$$= \frac{mv}{QB}\left[\cos\left(\frac{QB}{m}t\right) - 1\right],$$

where we have used the initial condition that $x = 0$ when $t = 0$.

(c) An ion hits the grid when $z = 0$. From the solution for z we see that $z = 0$ when $\sin QBt/m = 0$. Ignoring the $t = 0$ solution, the next time that the ion passes through the $z = 0$ plane is when $QBt/m = \pi$, or $t = \pi m/QB$. At this instant,

$$x = \frac{mv}{QB}(\cos\pi - 1) = -\frac{2mv}{QB},$$

and

$$y = \frac{QE}{2m}\left(\frac{\pi m}{QB}\right)^2 = \frac{\pi^2 mE}{2QB^2},$$

as required.

Note that the y*-displacement depends on the charge-to-mass ratio of the ion, whereas the* x*-displacement also depends on the initial speed.*

Solution 6.6 For fields with varying direction, the drift velocity is given by

$$\mathbf{v}_\mathrm{d} = \frac{mv_\parallel^2}{qBR_B}\,\widehat{\mathbf{R}}_B \times \widehat{\mathbf{B}},$$

where $\mathbf{R}_B$ is a vector drawn to a point on the field line from the centre of curvature of the line at that point. The field lines for a long straight current-carrying wire are circles centred on the wire. So the electron will spiral round a circular field line centred on the wire, which means that R_B will remain constant, and $\widehat{\mathbf{R}}_B \equiv \mathbf{e}_r$. The field produced by the wire is given by the standard expression (see Figure 5.2a of Book 2)

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\,\mathbf{e}_\phi,$$

so $\widehat{\mathbf{R}}_B \times \widehat{\mathbf{B}} = \mathbf{e}_r \times \mathbf{e}_\phi = \mathbf{e}_z$. So

$$\mathbf{v}_\mathrm{d} = \frac{mv_\parallel^2}{qBR_B}\,\mathbf{e}_z = \frac{mv_\parallel^2}{-eR_B}\frac{2\pi R_B}{\mu_0 I}\,\mathbf{e}_z = -\frac{2\pi m v_\parallel^2}{\mu_0 e I}\,\mathbf{e}_z.$$

For the drift due to the varying strength of the field we use Equation 6.20:

$$\mathbf{v}_\mathrm{d} = \frac{mv_\perp^2}{2qB^3}\,\mathbf{B} \times \operatorname{grad} B.$$

Since $B = \mu_0 I/2\pi r$, using the expression for grad in cylindrical coordinates,

$$\operatorname{grad} B = \frac{\mathrm{d}B_r}{\mathrm{d}r}\,\mathbf{e}_r = -\frac{\mu_0 I}{2\pi r^2}\,\mathbf{e}_r = -\frac{B}{r}\,\mathbf{e}_r.$$

Hence

$$\mathbf{B} \times \operatorname{grad} B = B\mathbf{e}_\phi \times \left(-\frac{B}{r}\,\mathbf{e}_r\right) = \frac{B^2}{r}\,\mathbf{e}_z.$$

So for an electron spiralling around a field line with radius R_B, the drift velocity is

$$\mathbf{v}_\mathrm{d} = \frac{mv_\perp^2}{-2eB^3}\frac{B^2}{R_B}\,\mathbf{e}_z = -\frac{mv_\perp^2}{2eBR_B}\,\mathbf{e}_z,$$

and substituting $B = \mu_0 I/2\pi R_B$ we obtain

$$\mathbf{v}_\mathrm{d} = -\frac{\pi m v_\perp^2}{\mu_0 e I}\,\mathbf{e}_z.$$

Note that since $v_\parallel = v_\perp$ in this case, the drift due to field curvature is twice that due to the gradient of field strength, but both contributions are in the $-\mathbf{e}_z$-direction.

The magnitude of the drift velocity is given by

$$\begin{aligned} v_\mathrm{d} &= 1.5 \times \frac{2\pi m v_\parallel^2}{\mu_0 e I} \\ &= \frac{1.5 \times 9.11 \times 10^{-31}\,\mathrm{kg} \times (1.0 \times 10^5\,\mathrm{m\,s^{-1}})^2}{2 \times 10^{-7}\,\mathrm{N\,A^{-2}} \times 1.60 \times 10^{-19}\,\mathrm{C} \times 100\,\mathrm{A}} \\ &= 4.3 \times 10^3\,\mathrm{m\,s^{-1}}. \end{aligned}$$

Note that the drift velocity is independent of R_B, the radius of the orbit about the current.

Solution 6.7 (a) The electron's speed perpendicular to the magnetic field initially is $v_\perp = v_\phi = 3.0 \times 10^6\,\mathrm{m\,s^{-1}}$. Its cyclotron radius is therefore

$$\begin{aligned} r_\mathrm{c} &= \frac{mv_\perp}{|q|B} \\ &= \frac{9.11 \times 10^{-31}\,\mathrm{kg} \times 3.0 \times 10^6\,\mathrm{m\,s^{-1}}}{1.60 \times 10^{-19}\,\mathrm{C} \times 2.0 \times 10^{-3}\,\mathrm{T}} \\ &= 8.5 \times 10^{-3}\,\mathrm{m}. \end{aligned}$$

(b) The quantity $(\sin^2\alpha)/B$ remains constant as the particle moves through the bottle. The initial value α_0 of the pitch angle is

$$\alpha_0 = \tan^{-1}(v_\phi/v_z) = \tan^{-1}(3/2) = 56.3^\circ,$$

and the initial field strength is $2.0 \times 10^{-3}\,\mathrm{T}$, so

$$\frac{\sin^2\alpha}{B} = \frac{\sin^2 56.3^\circ}{2.0 \times 10^{-3}\,\mathrm{T}} = 3.46 \times 10^2\,\mathrm{T^{-1}}.$$

At the mirror point, $\alpha = 90^\circ$ and $\sin^2\alpha = 1$, so

$$B = 1/(3.46 \times 10^2\,\mathrm{T^{-1}}) = 2.9 \times 10^{-3}\,\mathrm{T}.$$

(c) At the mirror point, $v_z = 0$. Since the electron's energy is conserved, $v_\perp$ must now be equal to the initial speed of the electron, that is,

$$v_\perp = \sqrt{3.0^2 + 2.0^2} \times 10^6\,\mathrm{m\,s^{-1}} = 3.6 \times 10^6\,\mathrm{m\,s^{-1}}.$$

So the radius of the electron's orbit at the mirror point is

$$\begin{aligned} r_\mathrm{c} &= \frac{mv_\perp}{|q|B} \\ &= \frac{9.11 \times 10^{-31}\,\mathrm{kg} \times 3.6 \times 10^6\,\mathrm{m\,s^{-1}}}{1.60 \times 10^{-19}\,\mathrm{C} \times 2.9 \times 10^{-3}\,\mathrm{T}} \\ &= 7.1 \times 10^{-3}\,\mathrm{m}. \end{aligned}$$

(d) The value of $\sin^2\alpha$ must be the equal to its initial value when the electron returns to the $z = 0$ plane, that is $\sin^2\alpha = \sin^2 56.3^\circ$. This has solutions $\alpha = 56.3^\circ,\ (180^\circ \pm 56.3^\circ),\ (360^\circ - 56.3^\circ)$. The appropriate solution, corresponding to the electron continuing to orbit in the same sense about the z-axis, but with the z-component of velocity reversed, is $\alpha = 123.7^\circ$, and the velocity is

$$\mathbf{v} = (3.0\,\mathbf{e}_\phi - 2.0\,\mathbf{e}_z) \times 10^6\,\mathrm{m\,s^{-1}}.$$

Book 2 Chapter 7

Solution 7.1 The planar symmetry indicates that the electric field and the current density are everywhere perpendicular to the plane of the slabs, and we take this to be the z-direction. In a steady state, the same current flows across any plane with constant z within the slabs, and therefore the current density, J_z, say, is the same at all points in the slabs. The electric fields within the two slabs, $E_{z1}\,\mathbf{e}_z$ and $E_{z2}\,\mathbf{e}_z$, are then given by

$$J_z = \sigma_1 E_{z1} = \sigma_2 E_{z2}.$$

The potential difference across the two slabs is (see Figure 14)

$$\Delta V = E_{z1} d_1 + E_{z2} d_2.$$

From the first equation, $E_{z2} = E_{z1}\sigma_1/\sigma_2$ and substituting this expression into the second equation leads to

$$E_{z1} = \frac{\Delta V}{d_1 + (\sigma_1/\sigma_2)d_2}.$$

Now the potential difference across slab 1 is $E_{z1}d_1$, and since the outer boundary of this slab is at potential $V = 0$, the potential at the interface is

$$\begin{aligned} V_{\text{inter}} &= E_{z1}d_1 \\ &= \frac{\Delta V}{d_1 + (\sigma_1/\sigma_2)d_2} d_1 \\ &= \frac{\Delta V}{1 + (\sigma_1 d_2/\sigma_2 d_1)}. \end{aligned}$$

If $d_1 \to 0$ or $d_2 \to \infty$, then $V_{\text{inter}} \to 0$.
If $d_1 \to \infty$ or $d_2 \to 0$, then $V_{\text{inter}} \to \Delta V$.
If $\sigma_1 \to 0$ or $\sigma_2 \to \infty$, then $V_{\text{inter}} \to \Delta V$.
If $\sigma_1 \to \infty$ or $\sigma_2 \to 0$, then $V_{\text{inter}} \to 0$.

All of these limiting values are in agreement with what would be expected.

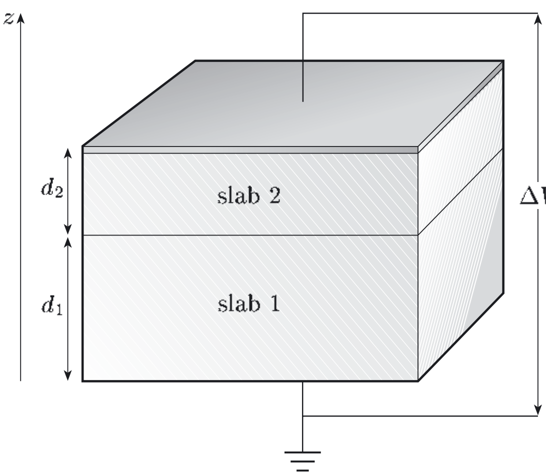

Figure 14 For Solution 7.1.

Solution 7.2 We denote the length, radius and wall thickness of the tube by l, r and t, respectively. For each pattern of current flow, resistance = 'length' / ('area' $\times$ conductivity), where 'length' is measured in the direction of current flow and 'area' is measured perpendicular to current flow. See Figure 15.

(a) For current flow along the length of the tube, 'length' $= l$, 'area' $= 2\pi r t$,

so

$$R_\text{a} = \frac{l}{2\pi r t \sigma} = \frac{0.1\,\text{m}}{2\pi \times 5.0 \times 10^{-3}\,\text{m} \times 5.0 \times 10^{-4}\,\text{m} \times 1.0 \times 10^{7}\,\Omega^{-1}\,\text{m}^{-1}} = 6.4 \times 10^{-4}\ \Omega.$$

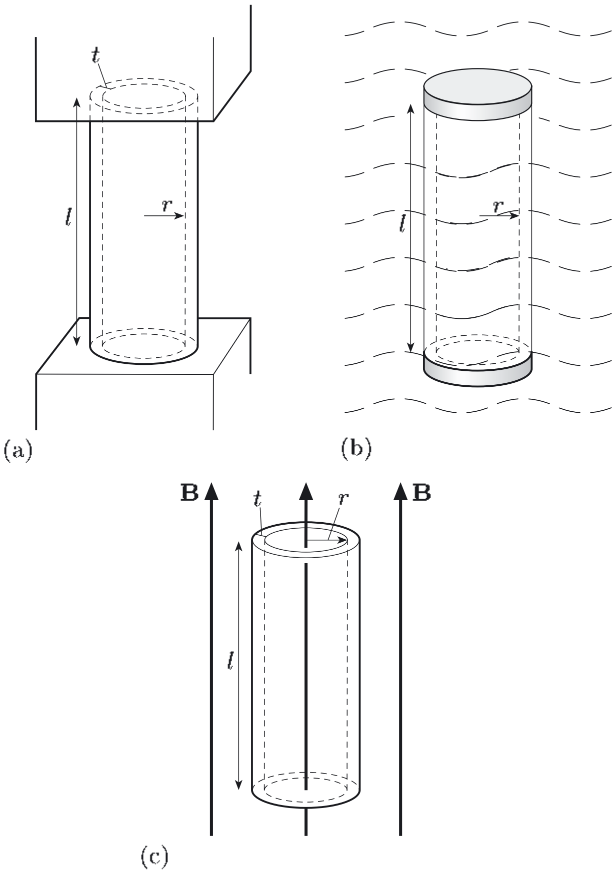

Figure 15 For Solution 7.2.

(b) For current flow between the inner and outer surfaces of the wall of the tube, 'length' $= t$, 'area' $= 2\pi r l$,

so

$$R_\text{b} = \frac{t}{2\pi r l \sigma} = \frac{5.0 \times 10^{-4}\,\text{m}}{2\pi \times 5.0 \times 10^{-3}\,\text{m} \times 0.1\,\text{m} \times 1.0 \times 10^{7}\,\Omega^{-1}\,\text{m}^{-1}} = 1.6 \times 10^{-8}\ \Omega.$$

(c) For current flow around the wall of the tube, 'length' $= 2\pi r$, 'area' $= lt$, so

$$R_\text{c} = \frac{2\pi r}{l t \sigma} = \frac{2\pi \times 5.0 \times 10^{-3}\,\text{m}}{0.1\,\text{m} \times 5.0 \times 10^{-4}\,\text{m} \times 1.0 \times 10^{7}\,\Omega^{-1}\,\text{m}^{-1}} = 6.3 \times 10^{-5}\ \Omega.$$

Solution 7.3 The equation above Figure 7.10 of Book 2 gives the potential along the perpendicular from the small sphere to the metal surface. The potential difference between the sphere and the surface is obtained by integrating this potential along the perpendicular line:

$$\begin{aligned}\Delta V &= \frac{I}{4\pi\sigma}\int_a^d \left(\frac{1}{z} - \frac{1}{2d-z}\right)\mathrm{d}z \\ &= \frac{I}{4\pi\sigma}\Big[\ln z + \ln(2d-z)\Big]_a^d \\ &= \frac{I}{4\pi\sigma}\left(\ln\frac{d}{a} + \ln\frac{d}{2d-a}\right) \\ &= \frac{I}{4\pi\sigma}\ln\left(\frac{d^2}{a(2d-a)}\right).\end{aligned}$$

The resistance is therefore

$$R = \frac{\Delta V}{I} = \frac{1}{4\pi\sigma}\ln\left(\frac{d^2}{a(2d-a)}\right).$$

For $d/a = 10$ and $\sigma = 5.0\,\Omega^{-1}\,\mathrm{m}^{-1}$,

$$R = \frac{1}{4\pi \times 5.0\,\Omega^{-1}\,\mathrm{m}^{-1}}\ln\frac{100}{19} = 2.6\times 10^{-2}\,\Omega.$$

Solution 7.4 As discussed in the solution to Exercise 7.6 of Book 2, the increasing field of the solenoid induces a current in the ring that creates a magnetic dipole in the opposite direction to the field of the solenoid. In the uniform field at the centre of the solenoid there is no net force on the dipole — just as there is no translational force on a bar magnet that is aligned with a uniform field. When the ring is at the end of the solenoid, the field around it is not uniform, and there is a net repulsive force, as described in the solution to the exercise, but the force is much smaller in the absence of the iron core.

The effect on the ring can be understood by considering the magnetic forces that act on the currents induced in the ring. The magnetic force acting on a current element $I\,\delta\mathbf{l}$ in a field $\mathbf{B}$ is $\mathbf{F} = I\,\delta\mathbf{l}\times\mathbf{B}$. When the ring is in the uniform axial field near the centre of the solenoid, the vector product of $I\,\delta\mathbf{l}$ (in a clockwise azimuthal direction viewed from above) and $\mathbf{B}$ (vertically upwards) is radially inwards at all points on the ring. So there is no net force on the ring when it is near the centre of the solenoid. At the end of the solenoid the field is not uniform; it decreases with distance and the field lines spread out — they have a component in the radial direction, away from the axis of the solenoid, in addition to the axial component. The vector product of $I\,\delta\mathbf{l}$ with this outward radial component gives an upwards force at each point on the ring. However, without the magnetic core, the force is likely to be too weak to propel the ring upwards.

Solution 7.5 The flux through the coil when the normal to the coil is at an angle θ to the direction of the horizontal part of the magnetic field is $\Phi = NAB_{\mathrm{horiz}}\cos\theta$. Note that the vertical component of the field doesn't contribute anything to the flux through the coil, since it is always parallel to the rotating coil. The angle θ varies as $\theta = 2\pi ft$. So

$$V_{\mathrm{emf}} = -\mathrm{d}\Phi/\mathrm{d}t = 2\pi fNAB_{\mathrm{horiz}}\sin 2\pi ft,$$

which has an amplitude

$$V_0 = 2\pi fNAB_{\mathrm{horiz}},$$

and hence

$$B_{\mathrm{horiz}} = V_0/2\pi fNA.$$

For the data quoted,

$$\begin{aligned}B_{\mathrm{horiz}} &= 25\times 10^{-3}\,\mathrm{V}/(2\pi\times 80\,\mathrm{Hz} \\ &\quad \times 50\times 0.50\times 10^{-4}\,\mathrm{m}^2) \\ &= 0.020\,\mathrm{T}.\end{aligned}$$

Solution 7.6 (a) The emf, V_{B}, induced in coil B, is given by

$$V_{\mathrm{B}} = -M\,\frac{\mathrm{d}I_{\mathrm{A}}}{\mathrm{d}t} = -\omega MI_{\mathrm{A0}}\cos\omega t.$$

Since $R_{\mathrm{B}} \gg \omega L_{\mathrm{B}}$, the current in coil B does not depend on the self-inductance, so

$$I_{\mathrm{B}} = \frac{V_{\mathrm{B}}}{R_{\mathrm{B}}} = -\frac{\omega MI_{\mathrm{A0}}}{R_{\mathrm{B}}}\cos\omega t.$$

Comparing this with the expression given for the current in coil B, we see that

$$\frac{\omega MI_{\mathrm{A0}}}{R_{\mathrm{B}}} = I_{\mathrm{B0}}, \quad \text{so} \quad M = \frac{I_{\mathrm{B0}}R_{\mathrm{B}}}{\omega I_{\mathrm{A0}}}.$$

(b) The value of the mutual inductance between the two circuits is the same when a current in B induces a current in A as when a current in A induces a current in B. So in this case,

$$V_{\mathrm{A}} = -M\,\frac{\mathrm{d}I_{\mathrm{B}}}{\mathrm{d}t} = \frac{I_{\mathrm{B0}}R_{\mathrm{B}}}{\omega I_{\mathrm{A0}}}\alpha I'_{\mathrm{B0}}\exp\left[-\alpha t\right].$$

Again, $I_{\mathrm{A}} = V_{\mathrm{A}}/R_{\mathrm{A}}$, so

$$I_{\mathrm{A}} = \frac{\alpha I'_{\mathrm{B0}}I_{\mathrm{B0}}R_{\mathrm{B}}}{\omega I_{\mathrm{A0}}R_{\mathrm{A}}}\exp\left[-\alpha t\right].$$

Solution 7.7 To determine L we need to find the flux through the coil for a given current, and to determine the flux we need to know the magnetic field in the core. We assume that the magnetic flux is confined to the core. Symmetry indicates that the magnetic intensity is in the azimuthal direction for cylindrical coordinates with their z-axis perpendicular to the plane of the disc and through its centre, and that its magnitude is constant around a circular path with radius r within the core. So applying Ampère's law to such a path,

$H(r)\,2\pi r = NI$, and
$B(r) = \mu\mu_0 H(r) = \mu\mu_0 NI/2\pi r.$

The flux through each turn is the same, so the total flux through the coil is N times the flux through a single turn. Within a single turn, the field is constant within a thin strip, of width δr that is parallel to the z-axis and at a distance r from it, and the field is perpendicular to the plane of the strip, so the flux through the strip is

$$\delta\Phi = B(r)\,a\,\delta r,$$

and the total flux through N turns is

$$\Phi = N\frac{\mu\mu_0 NI}{2\pi}\,a\int_a^{2a}\frac{1}{r}\,\mathrm{d}r = \frac{\mu\mu_0 N^2 Ia}{2\pi}\,\ln 2.$$

Thus $\quad L = \dfrac{\mathrm{d}\Phi}{\mathrm{d}I} = \dfrac{\mu\mu_0 N^2 a}{2\pi}\,\ln 2.$

Substituting the data given in the question,

$$L = \frac{500\times 4\pi\times 10^{-7}\,\mathrm{N\,A^{-2}}\times 10^4\times 4.0\times 10^{-3}\,\mathrm{m}}{2\pi}\,\ln 2$$
$$= 2.8\times 10^{-3}\,\mathrm{H}.$$

Book 2 Chapter 8

Solution 8.1 From Equation 8.4 of Book 2,

$$U = \tfrac{1}{2}\sum_{i=1}^{4} q_i V_i, \quad \text{where} \quad V_i = \sum_{j=1,\,j\neq i}^{4}\frac{q_j}{4\pi\varepsilon_0 r_{ij}}.$$

The symmetry of the problem indicates that the potentials V_i at the location of each of the charges are identical. For a particular charge i, the distances to the other three charges are $\sqrt{2}a$, $\sqrt{2}a$ and $2a$, so

$$V_i = 2\frac{q}{4\pi\varepsilon_0\sqrt{2}a} + \frac{q}{4\pi\varepsilon_0(2a)} = \frac{q}{4\pi\varepsilon_0 a}\left(\sqrt{2}+\tfrac{1}{2}\right).$$

Thus

$$U = \tfrac{1}{2}\times 4\times q\times\frac{q}{4\pi\varepsilon_0 a}\left(\sqrt{2}+\tfrac{1}{2}\right) = \frac{q^2}{4\pi\varepsilon_0 a}\left(2\sqrt{2}+1\right),$$

where the factor 4 takes account of the fact that there are 4 identical charges.

Solution 8.2 (a) $U = \frac{1}{2}CV^2$, so

$$C = \frac{2U}{V^2} = \frac{2\times 5.0\times 10^6\,\mathrm{J}}{(2.0\times 10^3\,\mathrm{V})^2} = 2.5\,\mathrm{F}.$$

(b) $U = \frac{1}{2}LI^2$, so

$$L = \frac{2U}{I^2} = \frac{2\times 5.0\times 10^6\,\mathrm{J}}{(2.0\times 10^3\,\mathrm{A})^2} = 2.5\,\mathrm{H}.$$

(c) For the RC circuit, time constant $= RC = 0.2\,\Omega\times 2.5\,\mathrm{F} = 0.5\,\mathrm{s}$.
For the RL circuit, time constant $= L/R = 2.5\,\mathrm{H}/0.2\,\Omega = 12.5\,\mathrm{s}$.

(d) Resonant frequency is

$$f_\mathrm{n} = \frac{\omega_\mathrm{n}}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{2.5\,\mathrm{H}\times 2.5\,\mathrm{F}}} = \frac{1}{2\pi\times 2.5\,\mathrm{s}} = 0.064\,\mathrm{Hz}.$$

Solution 8.3 The required ratio is the ratio of the electric and magnetic energy densities. For air, $\varepsilon = 1$ and $\mu = 1$, so

$$u_\mathrm{elec} = \tfrac{1}{2}\varepsilon_0 E^2 = 0.5\times 8.85\times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}(100\,\mathrm{V\,m^{-1}})^2 = 4.43\times 10^{-8}\,\mathrm{J}.$$

$$u_\mathrm{mag} = \frac{B^2}{2\mu_0} = \frac{(5.0\times 10^{-5}\,\mathrm{T})^2}{8\pi\times 10^{-7}\,\mathrm{N\,A^{-2}}} = 9.95\times 10^{-4}\,\mathrm{J}.$$

So the ratio is

$$\frac{u_\mathrm{elec}}{u_\mathrm{mag}} = \frac{4.43\times 10^{-8}\,\mathrm{J}}{9.95\times 10^{-4}\,\mathrm{J}} = 4.4\times 10^{-5},$$

or $\quad 4.4\times 10^{-5}\ :1.$

The magnetic field energy is clearly very much larger than the electric field energy.

Solution 8.4 (a) During the charging phase,

$$V_C = V_\mathrm{s}\left[1-\exp\left(-\frac{t}{RC}\right)\right].$$

At $t = 5\,\mathrm{s}$, $V_C/V_\mathrm{s} = 0.95$, so

$$\exp\left(-\frac{t}{RC}\right) = 0.05, \quad \text{and} \quad -\frac{t}{RC} = \ln 0.05.$$

Thus

$$R = -\frac{t}{C\ln 0.05} = -\frac{5\,\mathrm{s}}{1.0\times 10^{-3}\,\mathrm{F}\times\ln 0.05} = 1.7\,\mathrm{k\Omega}.$$

(b) Average power $\overline{P}$ in discharge time T is given by

$$\overline{P} = \frac{0.9\times\frac{1}{2}CV^2}{T} = \frac{0.9\times 0.5\times 1.0\times 10^{-3}\,\mathrm{F}\times(300\,\mathrm{V})^2}{2\times 10^{-3}\,\mathrm{s}} = 20\,\mathrm{kW}.$$

Solution 8.5 The charge will be distributed uniformly over the surface of the conducting sphere (since the charge density is zero inside a conductor). The symmetry then indicates that the electric field is spherically symmetric, with $\mathbf{E} = E_r(r)\,\mathbf{e}_r$.

Using the integral version of Gauss's law with a spherical Gaussian surface, radius r ($> R$),

$$E_r\times 4\pi r^2 = Q/\varepsilon_0, \quad \text{so} \quad E_r = Q/4\pi\varepsilon_0 r^2.$$

Alternatively, you could justify this result by saying that outside any spherically symmetric charge

distribution the field is the same as if the charge were all located at the centre of the distribution.

The total energy U stored in the field is obtained by integrating the energy density, $\frac{1}{2}\varepsilon_0 E^2$, over the region outside the sphere. Since the field strength is uniform within a thin spherical shell, radius r, thickness δr, volume $4\pi r^2\,\delta r$, the energy δU in such a shell is

$$\delta U = \tfrac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2\,\delta r,$$

and the total energy is

$$U = \frac{Q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{1}{r^2}\,\mathrm{d}r = \frac{Q^2}{8\pi\varepsilon_0 R},$$

as given in the question.

In Exercise 8.2 of Book 2 we showed that the potential energy of a conducting sphere with surface charge density σ is

$$U = \frac{2\pi\sigma^2 R^3}{\varepsilon_0},$$

or, since $\sigma = Q/4\pi R^2$,

$$U = \frac{Q^2}{8\pi\varepsilon_0 R},$$

which is the same as derived for the energy stored in the electric field.

This again indicates that there are two ways of thinking of the electric energy: we can think in terms of the energy of charges held at various potentials or we can think in terms of the energy stored in the electric fields in the regions around the charges. Both viewpoints lead to the same result for the energy stored in the system.

Solution 8.6 (a) Since the total charge Q in the sphere is related to the (uniform) charge density ρ by $Q = \frac{4}{3}\pi R^3\rho$, the expression for the potential energy derived in Worked Example 8.1 of Book 2 can be written as

$$U = \frac{4\pi R^5}{15\varepsilon_0}\left(\frac{3Q}{4\pi R^3}\right)^2 = \frac{3Q^2}{20\pi\varepsilon_0 R}.$$

(b) The electric field outside the sphere is the same as that outside the conducting sphere in the previous question — and the same as in the region $r > R$ around a point charge Q. So the energy stored in the electric field in this region is the same in all three cases, and given by the expression verified in Exercise 8.5.

$$U_{r>R} = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

(c) The difference between the expressions in (a) and (b) is

$$\Delta U = \frac{Q^2}{\pi\varepsilon_0 R}\left(\frac{3}{20} - \frac{1}{8}\right) = \frac{Q^2}{40\pi\varepsilon_0 R}.$$

This must be the energy stored in the electric field *inside* the sphere, that is, in the region $r < R$. In Worked Example 8.1 an expression was derived for the field inside the sphere: $E_r = \rho r/3\varepsilon_0$. Thus, using a similar integration method to the previous question, the energy stored in the field inside the sphere is

$$\begin{aligned} U_{r<R} &= \tfrac{1}{2}\varepsilon_0 \int_0^R \left(\frac{\rho r}{3\varepsilon_0}\right)^2 4\pi r^2\,\mathrm{d}r \\ &= \frac{2\pi\rho^2}{9\varepsilon_0}\int_0^R r^4\,\mathrm{d}r \\ &= \frac{2\pi\rho^2 R^5}{45\varepsilon_0}. \end{aligned}$$

Since $\rho = 3Q/4\pi R^3$,

$$U_{r<R} = \frac{2\pi R^5}{45\varepsilon_0}\left(\frac{3Q}{4\pi R^3}\right)^2 = \frac{Q^2}{40\pi\varepsilon_0 R},$$

confirming the suggestion that the difference between the energies in parts (a) and (b) is the energy stored in the electric field within the sphere.

Solution 8.7 The self-inductance of a solenoid with length l, area A and number of turns per unit length n is given by Equation 7.26 of Book 2:

$$\begin{aligned} L &= \mu_0 n^2 l A \\ &= 4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}} \times (200\,\mathrm{m^{-1}})^2 \\ &\quad \times 1.0\,\mathrm{m} \times \pi(0.10\,\mathrm{m})^2 = 1.6\,\mathrm{mH}. \end{aligned}$$

Thus the energy stored is

$$U = \tfrac{1}{2}LI^2 = 0.5\times 1.6\times 10^{-3}\,\mathrm{H}\times(100\,\mathrm{A})^2 = 7.9\,\mathrm{J}.$$

The power dissipated by Joule heating is given by $P = I^2 R$, where the resistance R is

$$\begin{aligned} R &= \frac{\text{wire length}}{\sigma \times \text{wire area}} \\ &= \frac{200 \times 2\pi \times 0.10\,\mathrm{m}}{6.5 \times 10^7\,\Omega^{-1}\,\mathrm{m}^{-1} \times \pi(2.5 \times 10^{-3}\,\mathrm{m})^2} \\ &= 9.8 \times 10^{-2}\,\Omega. \end{aligned}$$

So the power dissipated is

$$P = I^2 R = (100\,\mathrm{A})^2 \times 9.8 \times 10^{-2}\,\Omega = 980\,\mathrm{W}.$$

This is clearly not a practical proposition. The energy dissipated per second, 980 J, is over 100 times greater than the energy stored! Also, the energy stored is tiny — the amount required to lift a 1 kg bag of sugar through 0.8 m.

Book 2 Chapter 9

Solution 9.1 (a) The coil forms a continuous superconducting path, so the flux through it must remain constant when the magnetic field is reduced to zero, and the value of the flux must equal the flux through the coil due to the uniform applied field. This is simply

$$\Phi = NAB_0 = N\left(\pi a^2\right)B_0 = \pi N a^2 B_0.$$

(b) The self-inductance of the coil is defined by $L = \mathrm{d}\Phi/\mathrm{d}I$, so $L = \Phi/I$, and

$$I = \frac{\Phi}{L} = \frac{\pi N a^2 B_0}{L}.$$

(c) The field strength at the centre of a circular coil, with N turns of radius R, carrying current I, is $B_{\rm centre} = \mu_0 NI/2a$.
(*This can be deduced from the Biot–Savart law: a small element of length δl contributes $\delta B_{\rm centre} = \mu_0 I\,\delta l/4\pi a^2$, and since the fields due to all elements of the coil are in the same axial direction, the total field is just $N2\pi a/\delta l$ times greater than this.*) Thus

$$B_{\rm centre} = \frac{\mu_0 NI}{2a} = \frac{\mu_0 N}{2a}\frac{\pi N a^2 B_0}{L} = \frac{\mu_0 \pi N^2 a B_0}{2L}.$$

(d) The symmetry indicates that the field in the plane of the coil must be perpendicular to this plane. Because a point at $r = 0.95R$ is so close to the wire, the field there will be larger than at the centre. Note that though the field originally applied to the coil was uniform, it is the magnetic flux through the coil that must remain constant, not the magnetic field. The magnitude of the field near the centre will be smaller than that of the applied field B_0 and the field close to the turns of the coil will have a larger magnitude.

Solution 9.2 (a) From Table 9.1 of Book 2, $T_{\rm c} = 3.4\,{\rm K}$ for indium, and $B_{\rm c}(0) = 0.028\,{\rm T}$. The temperature T that corresponds to a critical field strength of $B_{\rm c}(T) = 0.014\,{\rm T}$ is given by Equation 9.1:

$$\frac{B_{\rm c}(T)}{B_{\rm c}(0)} = \frac{0.014\,{\rm T}}{0.028\,{\rm T}} = 0.5 = \left[1 - \left(\frac{T}{T_{\rm c}}\right)^2\right],$$

so

$$(T/T_{\rm c})^2 = 1 - 0.5 = 0.5, \quad \text{and}$$
$$T = \sqrt{0.5}T_{\rm c} = \sqrt{0.5} \times 3.4\,{\rm K} = 2.4\,{\rm K}.$$

(b) The critical current produces a magnetic field at the surface of the wire that is equal to the critical magnetic field. At a temperature of $1.2\,{\rm K}$, the critical field is given by

$$B_{\rm c}(1.2\,{\rm K}) = 0.028\,{\rm T}\left[1 - (1.2/3.4)^2\right] = 0.0245\,{\rm T}.$$

We can neglect the Earth's magnetic field, since it is very much smaller than the critical magnetic field. The magnetic field strength B at the surface of a wire of radius a carrying current I is given by $B = \mu_0 I/2\pi a$, so the critical current corresponding to a critical field of $0.0245\,{\rm T}$ is

$$I_{\rm c} = \frac{2\pi a B_{\rm c}}{\mu_0} = \frac{2\pi \times 1.0 \times 10^{-3}\,{\rm m} \times 0.0245\,{\rm T}}{4\pi \times 10^{-7}\,{\rm N\,A^{-2}}}$$
$$= 120\,{\rm A}.$$

Solution 9.3 Since $\xi_{\rm A} > \lambda_{\rm A}$, material A is a type-I superconductor, whereas for material B, $\xi_{\rm B} < \lambda_{\rm B}$ so it is a type-II material. In both cases, for very small applied magnetic fields, the field will deviate around the disc, as shown in Figure 9.21b in Book 2, and a superconducting screening current will flow around the perimeter of the disc. As the field is increased it eventually penetrates into the discs, destroying the superconductivity in the regions that it penetrates. For disc A, the magnetic field penetrates as 'swirling bands' (Figure 9.22), typically about a millimetre wide, with the proportion of the material that is in the normal state increasing as the magnetic field increases. For material B, the field penetrates as thin cylindrical cores, within which the material is normal. As the applied field increases, the number of normal cores increases and they become more closely packed together. Eventually a field is reached at which the cores merge and all of the material is normal.

Solution 9.4 Table contrasting type-I and type-II superconductors.

property	Type-I	Type-II
type of material	elements	alloys, ceramics, etc
critical temperature	low, generally $< 7\,\mathrm{K}$	alloys up to $30\,\mathrm{K}$, ceramics (HTS) up to $140\,\mathrm{K}$
refrigeration	liquid helium	liquid nitrogen for HTS
critical field	single critical field, $< 0.1\,\mathrm{T}$	lower and upper critical fields, upper B_c up to $20\,\mathrm{T}$
applications	few	many
ξ/λ	> 1	< 1
surface energy of NS boundary	positive, NS boundary area minimized	negative, NS boundary area maximized
field penetration	intermediate state, coarse layers of N and S, parallel to field, scale typically $\sim$ mm	mixed state, narrow cylindrical N cores, parallel to field, scale typically < 0.1 nm

Solution 9.5 (a) Assuming that the penetration depth λ is much smaller that the radius R of the cylinder, the field will fall exponentially with distance below the cylinder's surface, with characteristic decay distance λ:

$$B \propto \exp\left(\frac{r-R}{\lambda}\right) \quad \text{for } r \le R.$$

The boundary condition is that $H_{\|}$ is continuous at the surface, and since $\mu \approx 1$ for superconductors, this is equivalent to $B_{\|}$ being continuous at the surface. Thus at $r = R$ we require that $\mathbf{B} = B_{0z}\,\mathbf{e}_z$, and this boundary condition is satisfied by the expression

$$\mathbf{B} = B_{0z}\exp\left(\frac{r-R}{\lambda}\right)\mathbf{e}_z \quad \text{for } r \le R.$$

(b) Using Ampère's law,

$$\mu_0\mathbf{J} = \operatorname{curl}\mathbf{B} = \operatorname{curl}\left[B_z(r)\,\mathbf{e}_z\right] = -\frac{\partial B_z}{\partial r}\,\mathbf{e}_\phi.$$

So

$$\begin{aligned}\mathbf{J} &= -\frac{1}{\mu_0}\frac{\partial}{\partial r}\left(B_{0z}\exp\left(\frac{r-R}{\lambda}\right)\mathbf{e}_\phi\right)\\ &= -\frac{B_{0z}}{\mu_0\lambda}\exp\left(\frac{r-R}{\lambda}\right)\mathbf{e}_\phi.\end{aligned}$$

(c) The current flowing in a ring of radius r, thickness δr, per unit length of cylinder is $J_\phi(r)\,\delta r$, so the total screening current per unit length flowing near the surface is given by

$$\begin{aligned} i_\mathrm{s} &= \int_0^R J_\phi\,\mathrm{d}r = -\int_0^R \frac{B_{0z}}{\mu_0\lambda}\exp\left(\frac{r-R}{\lambda}\right)\mathrm{d}r\\ &= -\frac{B_{0z}}{\mu_0}\left[\exp\left(\frac{r-R}{\lambda}\right)\right]_0^R\\ &= -\frac{B_{0z}}{\mu_0}\left[1-\exp\left(\frac{-R}{\lambda}\right)\right] \simeq -\frac{B_{0z}}{\mu_0},\end{aligned}$$

since $R \gg \lambda$. The minus sign in the result indicates that the surface current circulates around the z-axis in the $-\mathbf{e}_\phi$-direction. This circulating current produces a uniform field inside the cylinder, parallel to the axis — it's equivalent to a solenoid with very closely wound turns. For a solenoid with n turns per unit length, carrying current I, the magnetic field inside the solenoid is $\mu_0 nI\,\mathbf{e}_z$, so replacing the current per unit length in this expression, nI, by i_s for the superconducting cylinder, the field produced by the screening current is given by

$$B_z = \mu_0 i_\mathrm{s} = \mu_0 \times \frac{-B_{0z}}{\mu_0} = -B_{0z},$$

which is equal in magnitude but opposite in direction to the applied field $\mathbf{B}_0 = B_{0z}\mathbf{e}_z$.

Solution 9.6 The flux through unit area aligned perpendicular to the field is $\Phi = BA = 0.50\,\mathrm{T\,m^2}$. Since each core contains one flux quantum ($2.07 \times 10^{-15}\,\mathrm{T\,m^2}$), the number of flux cores per unit area is

$$\frac{0.50\,\mathrm{T\,m^2}}{2.07 \times 10^{-15}\,\mathrm{T\,m^2}} = 2.4 \times 10^{14}\,,$$

that is, a number density of $2.4 \times 10^{14}\,\mathrm{m^{-2}}$.

The cores are arranged on a hexagonal lattice (Figure 9.23 of Book 2). If the nearest neighbour separation is a, then the area of an equilateral triangle of side length a is $\frac{1}{2}a^2\sin 60°$, and this is the area associated with one core. Thus

$$\tfrac{1}{2}a^2\sin 60° = \frac{1}{2.4 \times 10^{14}\,\mathrm{m^{-2}}},$$

so

$$a = \sqrt{\frac{2}{2.4 \times 10^{14}\,\mathrm{m^{-2}}\sin 60°}} = 9.8 \times 10^{-8}\,\mathrm{m}.$$

Book 2 Chapter 10

Solution 10.1 An invariant quantity has the same value in all inertial frames. Consider two frames, $\mathcal{F}$

and $\mathcal{F}'$, in standard configuration, with $\mathcal{F}'$ travelling at speed v in the $+x$-direction relative to frame $\mathcal{F}$. Let the electric and magnetic fields be (E_x, E_y, E_z) and (B_x, B_y, B_z) in frame $\mathcal{F}$ and (E'_x, E'_y, E'_z) and (B'_x, B'_y, B'_z) in frame $\mathcal{F}'$. Then using the field transformation equations (Equations 10.26–10.31 of Book 2),

$$\begin{aligned}
&E'^2 - c^2B'^2 \\
&= E'^2_x + E'^2_y + E'^2_z - c^2(B'^2_x + B'^2_y + B'^2_z) \\
&= E_x^2 + \gamma^2\left(E_y^2 - 2vE_yB_z + v^2B_z^2\right) \\
&\qquad + \gamma^2\left(E_z^2 + 2vE_zB_y + v^2B_y^2\right) \\
&- c^2\Bigg[B_x^2 + \gamma^2\left(B_y^2 + 2\frac{v}{c^2}B_yE_z + \frac{v^2}{c^4}E_z^2\right) \\
&\qquad\qquad + \gamma^2\left(B_z^2 - \frac{2v}{c^2}B_zE_y + \frac{v^2}{c^4}E_y^2\right)\Bigg].
\end{aligned}$$

The terms involving products of electric and magnetic field components all cancel, leaving

$$\begin{aligned}
&E'^2 - c^2B'^2 = (E_x^2 - c^2B_x^2) \\
&+ \gamma^2E_y^2\left(1 - \frac{v^2}{c^2}\right) + \gamma^2E_z^2\left(1 - \frac{v^2}{c^2}\right) \\
&- \gamma^2c^2B_y^2\left(1 - \frac{v^2}{c^2}\right) - \gamma^2c^2B_z^2\left(1 - \frac{v^2}{c^2}\right) \\
&= E_x^2 + E_y^2 + E_z^2 - c^2(B_x^2 + B_y^2 + B_z^2) \\
&= E^2 - c^2B^2.
\end{aligned}$$

A similar analysis could be used whatever the relative velocity of the two frames of reference, so $E^2 - c^2B^2$ has the same value in all inertial frames, that is, it is an invariant quantity — like the speed of light in free space, and electric charge.

Solution 10.2 (a) The electric field $\mathbf{E}$ in frame $\mathcal{F}$ is $(E,\,0,\,0)$ and the magnetic field $\mathbf{B}$ is $(B\cos\theta,\ B\sin\theta,\ 0)$. We use the field transformation equations (Equations 10.26–10.31 of Book 2) to find the fields in frame $\mathcal{F}'$:

$$\begin{aligned}
E'_x &= E_x = E, \\
E'_y &= \gamma(E_y - vB_z) = \gamma(0 - v\times 0) = 0, \\
E'_z &= \gamma(E_z + vB_y) = \gamma(0 + vB\sin\theta) = \gamma vB\sin\theta, \\
B'_x &= B_x = B\cos\theta \\
B'_y &= \gamma(B_y + vE_z/c^2) = \gamma(B\sin\theta + 0) = \gamma B\sin\theta, \\
B'_z &= \gamma(B_z - vE_y/c^2) = \gamma(0 - 0) = 0.
\end{aligned}$$

So the electric field in $\mathcal{F}'$ is $\mathbf{E}' = (E,\,0,\,\gamma vB\sin\theta)$, and the magnetic field is $\mathbf{B}' = (B\cos\theta,\,\gamma B\sin\theta,\,0)$. The angle θ' between the direction of $\mathbf{B}'$ and the x-axis is given by

$$\tan\theta' = \frac{\gamma B\sin\theta}{B\cos\theta} = \gamma\tan\theta.$$

(b) For $\theta' = 60°$, we require

$$\tan 60° = \gamma\tan 30°,$$

so $\gamma = \tan 60°/\tan 30° = 3.0$.

Hence

$$1 - \frac{v^2}{c^2} = \frac{1}{9}, \quad \text{and} \quad v = \sqrt{\frac{8}{9}}\,c = 2.83\times 10^8\,\mathrm{m\,s^{-1}}.$$

Solution 10.3 An invariant quantity has the same value in all inertial frames.

Rest mass is invariant; note that this mass is measured when the electron (or other particle) is at rest in a frame of reference. The relativistic mass, given by the ratio of the magnitude of the force acting on an electron to the magnitude of its acceleration, is not invariant but increases as the speed of the electron approaches the speed of light.

Charge is an invariant quantity, so the charge of an electron is the same in all inertial frames of reference.

The speed of an electron is *not* an invariant; the components of velocity transform as indicated in Equations 10.12–10.14 of Book 2.

The energy of an electron is *not* an invariant; it is given by $E = mc^2/\sqrt{1 - U^2/c^2}$, where m is the rest mass of the electron and U is the electron's speed in the frame of reference where the energy is measured.

The charge density is *not* invariant because the volume occupied by a specific amount of charge will vary between different inertial frames, as indicated by the Lorentz transformation of spatial coordinates.

The speed of light is invariant — this is one of the two basic postulates of special relativity.

The quantity $\mathbf{E}\cdot\mathbf{B}$ is invariant — see Worked Example 10.2.

Solution 10.4 (a) The positive charge is at rest, so there is no current density associated with this charge.

The negative charge travels at speed v in the $+x$-direction. In one second, charge $v\rho_-$ crosses unit area perpendicular to the $+x$-direction, so $J_{x-} = v\rho_-$. This is the only component of the current density — there are no components in the y- or z-directions.

(b) We will denote quantities in the frame that moves with the observer with primes. Then since this frame is in standard configuration with the frame in which the positive charge is at rest, the charge and current densities in it can be determined from Equations 10.22–10.25 of Book 2:

$$\begin{aligned}\rho'_- &= \gamma\left(\rho_- - \frac{v}{c^2}J_{x-}\right)\\ &= \gamma\left(\rho_- - \frac{v}{c^2}v\rho_-\right)\\ &= \gamma\rho_-\left(1-\frac{v^2}{c^2}\right),\\ \rho'_+ &= \gamma\left(\rho_+ - \frac{v}{c^2}J_{x+}\right)\\ &= \gamma\left(\rho_+ - \frac{v}{c^2}0\right)\\ &= \gamma\rho_+ = -\gamma\rho_-,\end{aligned}$$

$$\begin{aligned}J'_{x-} &= \gamma(J_{x-} - v\rho_-)\\ &= \gamma(v\rho_- - v\rho_-)\\ &= 0,\\ J'_{x+} &= \gamma(J_{x+} - v\rho_+)\\ &= \gamma(0 + v\rho_-) = \gamma J_{x-},\\ J'_{y-} &= J_{y-} = 0,\\ J'_{y+} &= J_{y+} = 0,\\ J'_{z-} &= J_{z-} = 0,\\ J'_{z+} &= J_{z+} = 0.\end{aligned}$$

$$\begin{aligned}\text{So, net charge density} &= \gamma\rho_-\left(1-\frac{v^2}{c^2}\right) - \gamma\rho_-\\ &= -\gamma\frac{v^2}{c^2}\rho_-,\end{aligned}$$

and the net current density only has an x-component, given by

$$J'_x = J'_{x+} = \gamma J_x = \gamma v\rho_-.$$

Solution 10.5 (a) Since charge Q is at rest in frame $\mathcal{F}$, there is no magnetic force on it. The electric force is deduced by first using Gauss's law to determine the electric field, which must be radially outward from the cylinder. For a Gaussian surface in the form of a cylinder with radius r, length l, coaxial with the cylinder of charge, the radial field at the surface has magnitude E given by

$$2\pi r l E = \rho A l/\varepsilon_0, \quad \text{so} \quad E = \rho A/2\pi\varepsilon_0 r.$$

The force on charge Q at $(0,\, d,\, 0)$ therefore has magnitude $F_{\text{elec}} = Q\rho A/2\pi\varepsilon_0 d$, and is in the $+y$-direction. This is the only force on Q observed in frame $\mathcal{F}$.

(b) The charge density ρ' and current density $\mathbf{J}'$ in frame $\mathcal{F}'$ are given by Equations 10.22–10.25 of Book 2. Thus

$$\rho' = \gamma\left(\rho - \frac{v}{c^2}J_x\right) = \gamma\rho,$$

since the current density J_x is zero in frame $\mathcal{F}$. Also

$$\begin{aligned}J'_x &= \gamma(J_x - v\rho)\\ &= -\gamma v\rho,\\ J'_y &= J_y = 0, \quad J'_z = J_z = 0.\end{aligned}$$

(c) The electric field in frame $\mathcal{F}'$ is deduced in the same way as in part (a), but using ρ' in place of ρ. Thus at distance r from the cylinder,

$$E' = \rho' A/2\pi\varepsilon_0 r = \gamma\rho A/2\pi\varepsilon_0 r,$$

and is directed radially outwards.

The current I' associated with current density J'_x is $I' = J'_x A$; note that since dimensions transverse to the direction of motion are unchanged, $A' = A$. At distance r from a long straight wire carrying current I' the magnetic field strength is

$$B' = \mu_0 I'/2\pi r = \mu_0 J'_x A/2\pi r = -\mu_0\gamma v\rho A/2\pi r,$$

with direction given by the right-hand grip rule.

(d) The test charge Q is moving with speed v in the $-x$-direction in frame $\mathcal{F}'$. The electric force on Q has magnitude

$$F'_{\text{elec}} = QE' = Q\gamma\rho A/2\pi\varepsilon_0 d,$$

and is in the $+y'$-direction.

The test charge moves perpendicularly to the azimuthal magnetic field, so the magnitude of the magnetic force at distance d from the cylinder is

$$F_{\text{mag}} = Qv|B'| = Qv\,\mu_0\gamma v\rho A/2\pi d = \mu_0 Q\gamma v^2\rho A/2\pi d.$$

Using the right-hand rule, with the velocity of Q in the $-x$-direction and the magnetic field circulating in the anticlockwise direction about the x-axis when viewed in the direction of increasing x, we deduce that this force is in the $-y$-direction.

The net force in the $+y$-direction is therefore

$$\begin{aligned}F' &= F'_{\text{elec}} - F'_{\text{mag}}\\ &= \frac{Q\gamma\rho A}{2\pi\varepsilon_0 d} - \frac{\mu_0 Q\gamma v^2\rho A}{2\pi d}\\ &= \frac{Q\gamma\rho A}{2\pi d}\left(\frac{1}{\varepsilon_0} - \mu_0 v^2\right).\end{aligned}$$

Now $c^2 = 1/\varepsilon_0\mu_0$, so $\mu_0 = 1/\varepsilon_0 c^2$, and the expression for the force can be rewritten as

$$F' = \frac{Q\gamma\rho A}{2\pi\varepsilon_0 d}\left(1-\frac{v^2}{c^2}\right) = \frac{Q\rho A}{2\pi\varepsilon_0\gamma d}.$$

(e) In both frames the force on charge Q is in the y-direction, so the equations for the x- and z-components are automatically satisfied and only Equation 10.18 of Book 2 is relevant. This equation indicates that the force in the frame where the particle is at rest is γ times the force in the frame where the particle is moving at speed v. In this question, the particle is at rest in frame $\mathcal{F}$, and so we expect $F_y = \gamma F'_y$. Comparison of the net forces obtained in parts (a) and (b) show that they are related in this way.

Note the entry for p. 227 of Book 2 in the list of errata for the course.

Solutions for Book 3

Book 3 Chapter 1

Solution 1.1 The general expression for the physical electric field of a plane wave is the real part of

$$\mathbf{E} = E_0 \exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)\right] \widehat{\mathbf{p}},$$

where E_0 is the amplitude, $\mathbf{k}$ is the propagation vector, ω is the angular frequency, ϕ is the phase shift and $\widehat{\mathbf{p}}$ is a unit vector in the polarization direction. In this case,

$$\begin{aligned}
\mathbf{k} &= \frac{k}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z),\\
\mathbf{k}\cdot\mathbf{r} &= \frac{k}{\sqrt{3}}\,(x+y+z),\\
\omega &= 2\pi f,\\
\widehat{\mathbf{p}} &= \frac{1}{\sqrt{2}}\,(\mathbf{e}_x - \mathbf{e}_y),\\
\phi &= 0,
\end{aligned}$$

and so

$$\mathbf{E} = \frac{E_0}{\sqrt{2}}\,\exp\left[\mathrm{i}\left(\frac{k}{\sqrt{3}}(x+y+z) - 2\pi f t\right)\right](\mathbf{e}_x - \mathbf{e}_y).$$

The physical electric field is the real part of this expression, that is

$$\mathbf{E}_{\text{phys}} = \frac{E_0}{\sqrt{2}}\,\cos\left(\frac{k}{\sqrt{3}}(x+y+z) - 2\pi f t\right)(\mathbf{e}_x - \mathbf{e}_y).$$

(b) For a transverse wave, the electric field is perpendicular to the propagation direction, so $\widehat{\mathbf{E}}\cdot\widehat{\mathbf{k}} = 0$. For the wave in part (a),

$$\widehat{\mathbf{E}} = \widehat{\mathbf{p}} = \frac{1}{\sqrt{2}}\,(\mathbf{e}_x - \mathbf{e}_y) \quad \text{and} \quad \widehat{\mathbf{k}} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z),$$

so

$$\widehat{\mathbf{E}}\cdot\widehat{\mathbf{k}} = \frac{1}{\sqrt{6}}\,(\mathbf{e}_x - \mathbf{e}_y)\cdot(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z) = 0,$$

and therefore the field is transverse to the propagation direction.

(c) Any polarization that satisfies $\widehat{\mathbf{p}}\cdot\widehat{\mathbf{k}} = 0$ is a possible solution. Examples are $\widehat{\mathbf{p}} = (-\mathbf{e}_x + \mathbf{e}_z)/\sqrt{2}$ and $\widehat{\mathbf{p}} = (\mathbf{e}_x - 3\mathbf{e}_y + 2\mathbf{e}_z)/\sqrt{14}$.

Solution 1.2 A general expression for the electric field of a plane monochromatic electromagnetic wave is

$$\mathbf{E} = E_0 \cos\left(\mathbf{k}\cdot\mathbf{r} \pm \omega t + \phi\right) \widehat{\mathbf{p}},$$

where E_0 is the amplitude, $|\mathbf{k}|$ is the wavenumber, $\omega = 2\pi f$ is the angular frequency, ϕ is the phase shift and $\widehat{\mathbf{p}}$ is a unit vector in the polarization direction. Now $\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$, and $|\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$, so the wavenumber is

$$\begin{aligned}
k &= \sqrt{(3\beta)^2 + (2\beta)^2 + (\beta)^2}\\
&= \sqrt{14}\,\beta\\
&= \sqrt{14}\times 6.0\times 10^{6}\,\mathrm{m}^{-1}\\
&= 2.2\times 10^{7}\,\mathrm{m}^{-1}.
\end{aligned}$$

$$\begin{aligned}
\omega &= 2\pi f = \gamma,\\
\text{so}\quad f &= \gamma/2\pi\\
&= 6.7\times 10^{15}\,\mathrm{s}^{-1}/2\pi\\
&= 1.1\times 10^{15}\,\mathrm{Hz}.
\end{aligned}$$

The amplitude E_0 of the wave is given by

$$\begin{aligned}
E_0 &= \alpha|(\mathbf{e}_x - 3\mathbf{e}_y)|\\
&= 2.0\times 10^{2}\,\mathrm{V\,m^{-1}} \times \sqrt{1^2 + 3^2}\\
&= 6.3\times 10^{2}\,\mathrm{V\,m^{-1}}.
\end{aligned}$$

The phase speed of this wave is

$$v = \frac{\omega}{k} = \frac{6.7\times 10^{15}\,\mathrm{s}^{-1}}{2.2\times 10^{7}\,\mathrm{m}^{-1}} = 3.0\times 10^{8}\,\mathrm{m\,s^{-1}},$$

which is consistent with the value of the speed of light in vacuum, $c = 3.00\times 10^{8}\,\mathrm{m\,s^{-1}}$.

Solution 1.3 Since $\mathbf{B} = \widehat{\mathbf{k}}\times\mathbf{E}/c$, the direction of $\mathbf{B}$ is the direction of $\widehat{\mathbf{k}}\times\widehat{\mathbf{E}}$, which is the direction of $(2\mathbf{e}_x + 2\mathbf{e}_y - \mathbf{e}_z)\times(\mathbf{e}_y + 2\mathbf{e}_z) = 5\mathbf{e}_x - 4\mathbf{e}_y + 2\mathbf{e}_z$. Expressing the direction as a unit vector, the magnetic field is in the direction

$$(5\mathbf{e}_x - 4\mathbf{e}_y + 2\mathbf{e}_z)/\sqrt{45}.$$

Solution 1.4 You may have recognized the types of some of these waves by inspecting the form of the expressions given and comparing them with the original wave. If so, a simple statement of the type of wave would be sufficient. However, below we include some explanation to justify the answers.

(a) The plane wave $E_0\exp[\mathrm{i}(kx-\omega t)]\,\mathbf{e}_y$ is identical to the original wave, that is, it has the same amplitude E_0, wavenumber k, angular frequency ω, phase shift (zero) and polarization $\mathbf{e}_y$ and travels in the $+x$-direction. The physical electric field of the combination is the wave

$$\mathrm{Re}\left\{2E_0\exp[\mathrm{i}(kx-\omega t)]\,\mathbf{e}_y\right\} = 2E_0\cos(kx-\omega t)\,\mathbf{e}_y,$$

which has twice the amplitude of the original wave, but otherwise is the same.

(b) The wave $E_0\exp[\mathrm{i}(kx-\omega t)]\,\mathbf{e}_z$ differs from the original wave in that it is polarized in the z-direction, at 90° to the polarization of the original. The combination is a linearly polarized wave,

$$\begin{aligned}
&\mathrm{Re}\left\{E_0\exp[\mathrm{i}(kx-\omega t)]\,(\mathbf{e}_y + \mathbf{e}_z)\right\}\\
&= E_0\cos(kx-\omega t)\,(\mathbf{e}_y + \mathbf{e}_z),
\end{aligned}$$

which has amplitude $\sqrt{2}E_0$ and is polarized in the direction $(\mathbf{e}_y + \mathbf{e}_z)/\sqrt{2}$ (see Equation 1.25 and Figure 1.11 in Book 3).

(c) This case is similar to (b), but the additional wave has twice the amplitude of the original wave. The combination is again a linearly polarized wave,

$$\begin{aligned}
&\mathrm{Re}\left\{ E_0 \exp[\mathrm{i}(kx - \omega t)]\,(\mathbf{e}_y + 2\mathbf{e}_z)\right\} \\
&= E_0 \cos(kx - \omega t)\,(\mathbf{e}_y + 2\mathbf{e}_z),
\end{aligned}$$

which has amplitude $\sqrt{5}E_0$ and is polarized in the direction $(\mathbf{e}_y + 2\mathbf{e}_z)/\sqrt{5}$.

(d) The wave $E_0 \exp[\mathrm{i}(kx - \omega t + \pi/2)]\,\mathbf{e}_z$ is polarized at 90° to the original wave and its phase its advanced by $\pi/2$. The physical electric field is

$$\begin{aligned}
&\mathrm{Re}\left\{ E_0 \exp[\mathrm{i}(kx - \omega t)]\,\mathbf{e}_y \right. \\
&\quad \left. + E_0 \exp[\mathrm{i}(kx - \omega t + \pi/2)]\,\mathbf{e}_z \right\} \\
&= E_0 \left[\cos(kx - \omega t)\,\mathbf{e}_y + \cos(kx - \omega t + \pi/2)\,\mathbf{e}_z\right] \\
&= E_0 \left[\cos(kx - \omega t)\,\mathbf{e}_y - \sin(kx - \omega t)\,\mathbf{e}_z\right],
\end{aligned}$$

which is a circularly polarized wave (see Subsection 1.3.5 and Figures 1.12 and 1.13).

(e) This case is similar to (d) except that the phase of the wave $E_0 \exp[\mathrm{i}(kx - \omega t - \pi/2)]\,\mathbf{e}_z$ is retarded by $\pi/2$ relative to the original rather than being advanced by $\pi/2$. The resultant field is

$$E_0 \left[\cos(kx - \omega t)\,\mathbf{e}_y + \sin(kx - \omega t)\,\mathbf{e}_z\right],$$

which is a circularly polarized wave with the electric field vector rotating in the opposite sense to case (d).

(f) The wave $2E_0 \exp[\mathrm{i}(kx - \omega t + \pi/2)]\,\mathbf{e}_z$ has twice the amplitude of the wave in (d), but otherwise is the same. The combination of this wave with the original gives a physical electric field

$$E_0 \left[\cos(kx - \omega t)\,\mathbf{e}_y - 2\sin(kx - \omega t)\,\mathbf{e}_z\right].$$

The polarization direction rotates in the same way as in (d), but the electric field is twice as large when the polarization is in the direction $\mathbf{e}_z$ as when it is in the direction $\mathbf{e}_y$; this is an elliptically polarized wave, rather than a circularly polarized wave.

(g) The wave $E_0 \exp[\mathrm{i}(kx - \omega t - \pi)]\,\mathbf{e}_y$ is identical to the original except that its phase is shifted by π. At any position and time, the fields of the two waves are equal in magnitude but opposite in sign, and so the two waves cancel — they destructively interfere.

(h) The wave $E_0 \exp[\mathrm{i}(kx + \omega t)]\,\mathbf{e}_y$ is identical to the original except that it travels in the opposite direction. The combination of the two waves leads to the field

$$\begin{aligned}
&\mathrm{Re}\left\{ E_0 \big(\exp[\mathrm{i}(kx - \omega t)] + \exp[\mathrm{i}(kx + \omega t)]\big)\,\mathbf{e}_y \right\} \\
&= \mathrm{Re}\left\{ E_0 \exp[\mathrm{i}kx]\big(\exp[-\mathrm{i}\omega t] + \exp[+\mathrm{i}\omega t]\big)\,\mathbf{e}_y \right\} \\
&= 2E_0 \cos kx \cos \omega t\,\mathbf{e}_y.
\end{aligned}$$

This is a standing wave, not a travelling wave. Note that for $kx = m\pi$, where m is any integer, the electric field is always zero. *Standing waves are discussed in Chapter 5 of Book 3 in the context of reflection of electromagnetic waves from conducting surfaces.*

(i) In this case the two waves differ by 1% in frequency and wavenumber. Their combined field is

$$\begin{aligned}
&\mathrm{Re}\left\{ E_0 \big(\exp[\mathrm{i}(kx - \omega t)] \right. \\
&\quad \left. + \exp[\mathrm{i}(1.01kx - 1.01\omega t)]\big) \right\} \mathbf{e}_y \\
&= E_0 \big(\cos(kx - \omega t) + \cos(1.01kx - 1.01\omega t)\big)\,\mathbf{e}_y \\
&= 2E_0 \cos(1.005kx - 1.005\omega t) \times \\
&\qquad \cos(0.005kx - 0.005\omega t)\,\mathbf{e}_y,
\end{aligned}$$

where we have used the trigonometric identity

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

to combine the two cosine terms. This is a wave that has approximately the same frequency and amplitude as the original (0.5% greater actually), but its amplitude is modulated by a more slowly varying sinusoidal waveform, with wavenumber $0.005k$ and angular frequency 0.005ω. *This is an example of the phenomenon of beats, which may be familiar in an audio context.*

Solution 1.5 (a) We first use the wavelength to determine ω and the components of $\mathbf{k}$.

$$\begin{aligned}
\omega &= 2\pi f \\
&= \frac{2\pi c}{\lambda} \\
&= \frac{2\pi \times 3.00 \times 10^8\,\mathrm{m\,s^{-1}}}{527 \times 10^{-9}\,\mathrm{m}} \\
&= 3.6 \times 10^{15}\,\mathrm{s^{-1}}.
\end{aligned}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{527 \times 10^{-9}\,\mathrm{m}} = 1.2 \times 10^7\,\mathrm{m^{-1}}.$$

Since the pulse is travelling in the $+y$-direction, $k_y = 1.2 \times 10^7\,\mathrm{m^{-1}}$, and $k_x = k_z = 0$.

To find the fields, we first deduce the power crossing unit area, and hence the time-average of the magnitude of the Poynting vector. This is directly related to the amplitude of the electric (and magnetic) fields.

The mean power in the pulse is $300\,\mathrm{J}/10^{-9}\,\mathrm{s} = 3.0 \times 10^{11}\,\mathrm{W}$, so the power crossing

unit area, which is the time-averaged value of the Poynting vector, $\overline{N}$, is

$$\overline{N} = \frac{3.0 \times 10^{11}\,\mathrm{W}}{\pi \times (0.5 \times 10^{-4}\,\mathrm{m})^2} = 3.8 \times 10^{19}\,\mathrm{W\,m^{-2}}.$$

From Equation 1.34 of Book 3, $\overline{N} = \frac{1}{2}\varepsilon_0 E_0^2 c$,

so

$$E_0 = \sqrt{\frac{2\overline{N}}{\varepsilon_0 c}} = \sqrt{\frac{2 \times 3.8 \times 10^{19}\,\mathrm{W\,m^{-2}}}{8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}} \times 3.00 \times 10^8\,\mathrm{m\,s^{-1}}}}$$

$$= 1.7 \times 10^{11}\,\mathrm{V\,m^{-1}}.$$

Since the polarization is in the x-direction, $E_{0x} = 1.7 \times 10^{11}\,\mathrm{V\,m^{-1}}$, and $E_{0y} = E_{0z} = 0$.

The magnetic field is related to the electric field by $\mathbf{B} = \widehat{\mathbf{k}} \times \mathbf{E}/c$ (Equation 1.16), so

$$\mathbf{B} = (\mathbf{e}_y \times \mathbf{e}_x)\,\frac{1.7 \times 10^{11}\,\mathrm{V\,m^{-1}}}{3.00 \times 10^8\,\mathrm{m\,s^{-1}}} = -5.7 \times 10^2\,\mathrm{T}\;\mathbf{e}_z.$$

Thus $B_{0z} = -5.7 \times 10^2\,\mathrm{T}$, and $B_{0x} = B_{0y} = 0$.

(b) The peak value of the electric field is almost six orders of magnitude greater than required to ionize air. So the energy of the pulse would be absorbed rapidly in air, which is why beams with such high intensities are made to travel through a vacuum. When powerful beams must propagate in air, they are expanded with lenses, so that the Poynting vector has a much smaller magnitude.

Solution 1.6 A polaroid filter transmits the component of the electric field that is parallel to the axis of polarization and blocks the perpendicular component. The vertical electric field of the incident wave can be resolved into component fields $E_0 \cos 30°$ parallel to the axis of polarization of the first filter and $E_0 \sin 30°$ perpendicular to the axis of polarization of this filter. Only the parallel component is transmitted, so the amplitude of the wave incident on the second filter is $E_0 \cos 30°$. This wave, which is polarized at 30° to the vertical, can be resolved into components parallel and perpendicular to the axis of polarization of the second filter. In case (a) the axis is vertical, and the component of the wave in that direction is $(E_0 \cos 30°) \times \cos 30° = E_0 \cos^2 30° = 3E_0/4$, so this is the amplitude of the transmitted wave. In case (b) the axis of polarization of the second filter is horizontal, and the component of the wave in that direction is $(E_0 \cos 30°) \times \sin 30° = E_0 \sin 30° \cos 30° = \sqrt{3}E_0/4$, so this is the amplitude of the transmitted wave.

Note that even though the initial wave was vertically polarized, after passing through the first filter the wave has components of polarization in both vertical and horizontal directions.

Solution 1.7 (a) For a monochromatic plane wave, $B_0 = E_0/c$,

so

$$\overline{N} = \tfrac{1}{2}\varepsilon_0 E_0^2 c = \tfrac{1}{2}\varepsilon_0 B_0^2 c^3 = \tfrac{1}{2} B_0^2 c/\mu_0,$$

where we have made use of the relationship $c^2 = 1/\varepsilon_0\mu_0$.

(b) The magnitude of the Poynting vector is equal to the power per unit area,

so

$$B_0 = \left(\frac{2\overline{N}\mu_0}{c}\right)^{1/2}$$

$$= \left(\frac{2 \times 10^{-10}\,\mathrm{W\,m^{-2}} \times 4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}}}{3.00 \times 10^8\,\mathrm{m\,s^{-2}}}\right)^{1/2}$$

$$= 9.2 \times 10^{-13}\,\mathrm{T}.$$

Book 3 Chapter 2

Solution 2.1 For a Hertzian dipole, the dipole length δl is much less than the wavelength λ of the radiation it produces (Section 2.2 of Book 3). For the field to be well-approximated by the radiation terms alone, we require $r \gg \lambda/2\pi$ (Equation 2.18).

(a) $\lambda = c/f = 3 \times 10^8\,\mathrm{m\,s^{-1}}/50\,\mathrm{s^{-1}} = 6 \times 10^6\,\mathrm{m}$, which is much greater than the dipole length, 10 km. So this is a Hertzian dipole. The field is a radiation field for distance much greater than $6 \times 10^6\,\mathrm{m}/2\pi$, that is beyond 10^7 m, say.

(b) This dipole radiates VHF radiowaves, with $\lambda = 3 \times 10^8\,\mathrm{m\,s^{-1}}/10^8\,\mathrm{s^{-1}} = 3\,\mathrm{m}$, which is *shorter* than the dipole length, 10 m. So this is *not* a Hertzian dipole.

(c) This is a microwave source, with $\lambda = 3 \times 10^8\,\mathrm{m\,s^{-1}}/3 \times 10^9\,\mathrm{s^{-1}} = 0.1\,\mathrm{m}$, which is much greater than the dipole length, 1 cm. So this is a Hertzian dipole. The field is a radiation field for distance much greater than $0.1\,\mathrm{m}/2\pi$, that is beyond 20 cm, say.

Solution 2.2 When r is small the terms involving r^{-3} will be large compared with the terms involving r^{-1} and r^{-2}, so the electric field in Equation 2.17 reduces to the expression that is printed below it in Book 3:

$$\mathbf{E} = \frac{c^2\eta}{-\mathrm{i}\omega} \exp\left[\mathrm{i}(kr - \omega t)\right] \left(\frac{2\cos\theta}{r^3}\,\mathbf{e}_r + \frac{\sin\theta}{r^3}\,\mathbf{e}_\theta\right).$$

Now $\eta = \mu_0 I_0\,\delta l/4\pi$, and the strength of the current dipole $I_0\,\delta l$ is related to the strength of the charge dipole p_0 by $I_0\,\delta l = \omega p_0$. Substituting these into the pre-exponential term in the equation for $\mathbf{E}$ we obtain

$$\frac{c^2\eta}{-\mathrm{i}\omega} = \frac{c^2\mu_0 I_0\,\delta l}{-\mathrm{i}\,4\pi\omega} = \frac{c^2\mu_0\omega p_0}{-\mathrm{i}\,4\pi\omega} = \mathrm{i}\,\frac{c^2\mu_0 p_0}{4\pi}.$$

Then since $c^2 = 1/\varepsilon_0\mu_0$, this pre-exponential term becomes

$$\frac{\mathrm{i}\,p_0}{4\pi\varepsilon_0}.$$

In the limit $\omega \to 0$, the wavenumber $k = \omega/c \to 0$, and so the exponential term tends to unity. The physical electric field then becomes

$$\mathbf{E}_{\text{phys}} = \mathrm{Re}\left\{\frac{\mathrm{i}\,p_0}{4\pi\varepsilon_0}\left(\frac{2\cos\theta}{r^3}\,\mathbf{e}_r + \frac{\sin\theta}{r^3}\,\mathbf{e}_\theta\right)\right\}$$
$$= \frac{p_0}{4\pi\varepsilon_0}\left(\frac{2\cos\theta}{r^3}\,\mathbf{e}_r + \frac{\sin\theta}{r^3}\,\mathbf{e}_\theta\right),$$

which is the same as Equation 2.1 for the electrostatic field of a charge dipole.

Solution 2.3 (a) The wavelength of the radiation is $\lambda = c/f = 3\times10^8\,\mathrm{m\,s^{-1}}/3\times10^9\,\mathrm{s^{-1}} = 10\,\mathrm{cm}$, so $r \gg \lambda/2\pi$, which means that the fields are given by the expressions for the radiation fields in Equations 2.19 and 2.20.

The electric field produced by the Hertzian dipole is always in the $\mathbf{e}_\theta$ direction (though this direction depends on the location of the point where the field is measured). The magnitude of the amplitude is $\omega\eta\sin\theta/r$ so

$$[\text{amplitude at }(2.0\,\mathrm{m},\ \pi/4,\ \pi/2)]$$
$$= \frac{(\sin(\pi/4)/2.0\,\mathrm{m})}{(\sin(\pi/2)/1.0\,\mathrm{m})}[\text{amplitude at }(1.0\,\mathrm{m},\ \pi/2,\ 0)]$$
$$= \frac{1/(2.0\sqrt{2}\,\mathrm{m})}{1.0\,\mathrm{m}^{-1}}\,2.0\times10^{-6}\,\mathrm{V\,m^{-1}}$$
$$= 7.1\times10^{-7}\,\mathrm{V\,m^{-1}}.$$

So $\mathbf{E}_0 = 7.1\times10^{-7}\,\mathrm{V\,m^{-1}}\,\mathbf{e}_\theta$.

(b) The magnetic field at $(2.0\,\mathrm{m},\ \pi/4,\ \pi/2)$ is in the $\mathbf{e}_\phi$ direction. The magnitude of its amplitude is just the magnitude of the electric field amplitude divided by c, that is,

$$7.1\times10^{-7}\,\mathrm{V\,m^{-1}}/3.00\times10^{8}\,\mathrm{m\,s^{-1}} = 2.4\times10^{-15}\,\mathrm{T}.$$

So $\mathbf{B}_0 = 2.4\times10^{-15}\,\mathrm{T}\,\mathbf{e}_\phi$. (*Remember that the directions of the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ in spherical coordinates depend on the location of the point that is being considered.*)

Solution 2.4

From Equation 2.24 of Book 3, the time-averaged Poynting vector is given by

$$\overline{\mathbf{N}} = \frac{\omega^2\eta^2}{2c\mu_0 r^2}\sin^2\theta\,\mathbf{e}_r.$$

The distance r from the origin to the point $(2.0,\ -2.0,\ 1.0)$ m is $\sqrt{2.0^2+2.0^2+1.0^2} = 3.0\,\mathrm{m}$. The angle θ is the angle between the direction of the dipole and the direction of the position vector of the measurement point, and since the dipole is aligned with the y-axis, this is given by $\cos\theta = y/r = -2.0/3.0$, so $\theta = 132°$. Also, $\eta = \mu_0 I_0\,\delta l/4\pi$. Thus

$$\overline{\mathbf{N}} = \frac{\omega^2\mu_0(I_0\,\delta l)^2}{32\pi^2 c r^2}\sin^2 132°\,\mathbf{e}_r =$$
$$\frac{(2.0\times10^9\,\mathrm{s^{-1}})^2\times4\pi\times10^{-7}\,\mathrm{N\,A^{-2}}\times(3.0\times10^{-8}\,\mathrm{A\,m})^2}{32\pi^2\times3.00\times10^8\,\mathrm{m\,s^{-1}}\times(3.0\,\mathrm{m})^2}$$
$$\times\sin^2 132°\,\mathbf{e}_r$$
$$= 2.9\times10^{-15}\,\mathrm{W\,m^{-2}}\,\mathbf{e}_r,$$

where the unit vector $\mathbf{e}_r$ that indicates the direction of power flow is in the direction of the position vector of the measurement point, that is,

$$\mathbf{e}_r = (2.0\mathbf{e}_x - 2.0\mathbf{e}_y + 1.0\mathbf{e}_z)/3.0.$$

Solution 2.5

The power per unit area transmitted as radiation passes through a scattering medium is given by

$$\overline{N} = \overline{N_0}\exp[-n\sigma z] = \overline{N_0}\exp[-z/L_{\text{scat}}],$$

where $\overline{N_0}$ is the incident power per unit area, n is the number density of air molecules, σ is the mean scattering cross-section for air molecules and L_{scat} is the scattering length. The scattered power is $(\overline{N_0} - \overline{N})$.

The number density is given by the ideal gas law (Equation 2.33):

$$n = \frac{P}{k_{\mathrm{B}}T}$$
$$= \frac{1.0\times10^5\,\mathrm{Pa}}{1.38\times10^{-23}\,\mathrm{J\,K^{-1}}\times273\,\mathrm{K}}$$
$$= 2.65\times10^{25}\,\mathrm{m^{-3}}.$$

Thus the transmitted power after $1.0\,\mathrm{km}$ is

$$\overline{N} = \overline{N_0}\times$$
$$\exp[-2.65\times10^{25}\,\mathrm{m^{-3}}\times1.50\times10^{-30}\,\mathrm{m^2}\times10^3\,\mathrm{m}]$$
$$= 0.961\overline{N_0}.$$

Therefore the percentage of the incident power that is scattered is

$$(1-0.961)\times100\% = 3.9\%.$$

The scattering cross-section is proportional to ω^4 and therefore proportional to λ^{-4}. (We ignore the weak dependence of the polarizability on ω.) For red light with $\lambda = 600\,\mathrm{nm}$,

$$\sigma_{\text{red}} = \sigma_{\text{blue}}\left(\frac{\lambda_{\text{blue}}}{\lambda_{\text{red}}}\right)^4$$
$$= 1.50\times10^{-30}\,\mathrm{m^2}\left(\frac{400}{600}\right)^4$$
$$= 2.96\times10^{-31}\,\mathrm{m^2}.$$

Hence

$$\overline{N} =$$
$$\overline{N_0}\exp[-2.65\times10^{25}\,\mathrm{m^{-3}}\times2.96\times10^{-31}\,\mathrm{m^2}\times10^3\,\mathrm{m}]$$
$$= 0.992\overline{N_0}.$$

Therefore the percentage of the incident power that is scattered is

$$(1 - 0.992) \times 100\% = 0.8\%,$$

which is five times smaller than for blue light.

Solution 2.6

Light from the sky that reaches you after travelling perpendicular to the direction of the incident sunlight is most strongly polarized, with the polarization perpendicular to the plane containing the Sun, the region of the sky that you are observing and you (see Figure 2.16 in Book 3). When the Sun is directly overhead, with direct sunlight incident in the vertical direction, the light reaching you from the horizon in any direction will be strongly polarized in the horizontal direction. Light from the sky in directions close to the Sun will be unpolarized. Thus if you are holding the polarizing filter so that it blocks light that is polarized in the north–south direction, then the horizon will appear bright to the east and to the west, but the sky to the north and south will appear dark. Conversely, if the polarizing filter is rotated so that it blocks light that is polarized in the east–west direction, then the horizon will appear bright to the north and to the south, but the sky to the east and west will appear dark. However, a clearer way of describing what you would observe is to say that when you look towards the horizon in any direction, the light will be polarized in the horizontal direction. When you look close to the direction of the sunshade overhead, the light is unpolarized.

Solution 2.7 (a) The time-average power per unit area radiated by a Hertzian dipole is given by the magnitude of the Poynting vector (Equation 2.24 of Book 3):

$$\overline{N} = \frac{\omega^2\eta^2}{2c\mu_0 r^2}\sin^2\theta.$$

The ratio of the magnitudes of the Poynting vectors at the points $(50\,\mathrm{m},\,90°,\,0)$ and $(100\,\mathrm{m},\,60°,\,90°)$ is

$$\frac{\sin^2 90°/(50\,\mathrm{m})^2}{\sin^2 60°/(100\,\mathrm{m})^2} = \frac{16}{3} = 5.3.$$

Note that the magnitude of the Poynting vector is axially symmetric — it doesn't depend on the ϕ-coordinate.

(b) The average power radiated in the range $70° < \theta < 110°$ is given by (see Subsection 2.4.3)

$$\begin{aligned}\overline{W} &= \int_{70°}^{110°}\int_0^{2\pi}\frac{\omega^2\eta^2}{2c\mu_0 r^2}\sin^2\theta\, r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\\ &= \frac{\omega^2\eta^2}{2c\mu_0}\int_0^{2\pi}\mathrm{d}\phi\int_{70°}^{110°}\sin^3\theta\,\mathrm{d}\theta\\ &= \frac{\omega^2\eta^2}{2c\mu_0}(2\pi)\left[\tfrac{1}{3}\cos^3\theta - \cos\theta\right]_{70°}^{110°}\\ &= \frac{\omega^2\eta^2}{2c\mu_0}(2\pi)\,0.657.\end{aligned}$$

The total power radiated is determined in a similar way, except the θ integral runs from zero to 180° (or π), and this replaces the factor of 0.657 by 1.33 (as shown in Subsection 2.4.3). So the fraction of the total power radiated in the range $70° < \theta < 110°$ is $0.657/1.33 = 0.49$.

Book 3 Chapter 3

Solution 3.1 (a) $f = \omega/2\pi = 1.0 \times 10^8\,\mathrm{Hz}$. The frequency *does not* change when the radiation travels through different materials.

(b) Phase speed is given by $v = c/n$, where the refractive index n of the material is the square root of the relative permittivity ε. Thus $v = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}/\sqrt{4.0} = 1.5 \times 10^8\,\mathrm{m\,s^{-1}}$.

(c) $\lambda = c/nf = v/f = 1.5 \times 10^8\,\mathrm{m\,s^{-1}}/1.0 \times 10^8\,\mathrm{Hz} = 1.5\,\mathrm{m}$.

(d) Wavenumber $k = 2\pi/\lambda = 2\pi/1.5\,\mathrm{m} = 4.2\,\mathrm{m^{-1}}$.

Solution 3.2 (a) The fields for the incident, reflected and transmitted waves are

$$\begin{aligned}\mathbf{E}_\mathrm{i} &= E_0\exp[\mathrm{i}(ky - \omega t)]\,\mathbf{e}_z, \qquad \text{(given)}\\ \mathbf{B}_\mathrm{i} &= \frac{E_0}{c}\exp[\mathrm{i}(ky - \omega t)]\,\mathbf{e}_x,\\ \mathbf{E}_\mathrm{r} &= r_\mathrm{n}E_0\exp[\mathrm{i}(-ky - \omega t)]\,\mathbf{e}_z,\\ \mathbf{B}_\mathrm{r} &= -\frac{r_\mathrm{n}E_0}{c}\exp[\mathrm{i}(-ky - \omega t)]\,\mathbf{e}_x,\\ \mathbf{E}_\mathrm{t} &= t_\mathrm{n}E_0\exp[\mathrm{i}(nky - \omega t)]\,\mathbf{e}_z,\\ \mathbf{B}_\mathrm{t} &= \frac{t_\mathrm{n}\sqrt{\varepsilon}E_0}{c}\exp[\mathrm{i}(nky - \omega t)]\,\mathbf{e}_x.\end{aligned}$$

In writing down these expressions we have used the relationships

$$B_0 = \frac{E_0}{v} = \frac{nE_0}{c} = \frac{\sqrt{\varepsilon}E_0}{c},$$

$$E_\mathrm{r} = r_\mathrm{n}E_0 \quad \text{and} \quad E_\mathrm{t} = t_\mathrm{n}E_0.$$

We assume that all of the electric fields are in the same direction, and determined the directions of the magnetic fields from the relationship $\widehat{\mathbf{B}} = \widehat{\mathbf{k}} \times \widehat{\mathbf{E}}$.

(b) The boundary conditions are that $E_\parallel$ and $H_\parallel$ are the same on both sides of the boundary. Since $\mu = 1$, the second condition is equivalent to $B_\parallel$ being the same on both sides of the boundary. Thus

$$\mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{r} = \mathbf{E}_\mathrm{t} \quad \text{so} \quad 1 + r_\mathrm{n} = t_\mathrm{n},$$

and

$$1 - r_\mathrm{n} = \sqrt{\varepsilon}\,t_\mathrm{n}.$$

Solving these simultaneous equations gives

$$r_\mathrm{n} = \frac{1 - \sqrt{\varepsilon}}{1 + \sqrt{\varepsilon}} \quad \text{and} \quad t_\mathrm{n} = \frac{2}{1 + \sqrt{\varepsilon}}.$$

These expressions are equivalent to those in Equation 3.14 of Book 3 if we set $n_1 = 1$ and $n_2 = \sqrt{\varepsilon}$.

Solution 3.3 (a) The refractive index of a material is $n = c/v = (\varepsilon\mu)^{1/2}$. For dielectric materials, $\mu = 1$ generally, so $n = \varepsilon^{1/2} = 1.7^{1/2} = 1.304$, or 1.3 to two significant figures.

For normal incidence in air, the expression for the reflectance in Equation 3.17 of Book 3 becomes

$$R = \left(\frac{1-n}{1+n}\right)^2 = \left(\frac{1-1.304}{1+1.304}\right)^2 = 0.017.$$

(b) The amplitudes of the reflected and transmitted waves are obtained from the amplitude reflection ratios (Equations 3.14):

$$\begin{aligned} E_{0\mathrm{r}} &= E_{0\mathrm{i}}\left(\frac{1-n}{1+n}\right) \\ &= 5.0\,\mathrm{V\,m^{-1}}\left(\frac{1-1.304}{1+1.304}\right) \\ &= -0.66\,\mathrm{V\,m^{-1}}; \end{aligned}$$

$$\begin{aligned} E_{0\mathrm{t}} &= E_{0\mathrm{i}}\left(\frac{2}{1+n}\right) \\ &= 5.0\,\mathrm{V\,m^{-1}}\left(\frac{2}{1+1.304}\right) \\ &= 4.34\,\mathrm{V\,m^{-1}}. \end{aligned}$$

The negative amplitude for $E_{0\mathrm{r}}$ indicates that the reflected wave is π out of phase with respect to the incident wave.

The boundary condition at the interface is $\mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{r} = \mathbf{E}_\mathrm{t}$, which in this case reduces to $E_{0\mathrm{i}} + E_{0\mathrm{r}} = E_{0\mathrm{t}}$. Substituting numerical values for the left-hand side,

$$5.0\,\mathrm{V\,m^{-1}} - 0.66\,\mathrm{V\,m^{-1}} = 4.34\,\mathrm{V\,m^{-1}},$$

which is equal to the calculated amplitude of the transmitted wave, as required.

Solution 3.4 The transmittance — the fraction of the power transmitted across a boundary — is independent of the direction that radiation travels across the boundary (see Equation 3.17 of Book 3). So the solutions for part (a) and part (b) are the same. For light travelling from air to glass, $n_1 = 1.00$ and $n_2 = 1.50$, so the transmittance of the air–glass boundary is

$$\begin{aligned} T_{\mathrm{air-glass}} &= \frac{4n_1n_2}{(n_1+n_2)^2} \\ &= \frac{4\times 1.00\times 1.50}{(1.00+1.50)^2} \\ &= 0.960. \end{aligned}$$

For the glass–water boundary,

$$T_{\mathrm{glass-water}} = \frac{4\times 1.50\times 1.33}{(1.50+1.33)^2} = 0.996.$$

So the transmittance through the glass wall from air into water is

$$T_{\mathrm{glass\,wall}} = 0.960 \times 0.996 = 0.957,$$

or 0.96 to two significant figures.

Note that the transmittance tends to unity as the difference between the refractive indices on either side of the boundary tends to zero.

Solution 3.5 The incident beam, which is polarized at 45° to the scattering plane, can be resolved into two components, one normal to the scattering plane and one in the scattering plane, with equal amplitudes and equal powers. For each component we determine the reflectance using the expressions in Equations 3.29 and 3.33 of Book 3, which require knowledge of θ_t. From Snell's law,

$$\theta_\mathrm{t} = \sin^{-1}(\sin 45^\circ/2.42) = 16.7^\circ.$$

Thus

$$\begin{aligned} R_\mathrm{isp} &= \left(\frac{n_1\cos\theta_\mathrm{t} - n_2\cos\theta_\mathrm{i}}{n_1\cos\theta_\mathrm{t} + n_2\cos\theta_\mathrm{i}}\right)^2 \\ &= \left(\frac{\cos 16.7^\circ - 2.42\cos 45^\circ}{\cos 16.7^\circ + 2.42\cos 45^\circ}\right)^2 \\ &= 0.080. \end{aligned}$$

$$\begin{aligned} R_\mathrm{nsp} &= \left(\frac{n_1\cos\theta_\mathrm{i} - n_2\cos\theta_\mathrm{t}}{n_1\cos\theta_\mathrm{i} + n_2\cos\theta_\mathrm{t}}\right)^2 \\ &= \left(\frac{\cos 45^\circ - 2.42\cos 16.7^\circ}{\cos 45^\circ + 2.42\cos 16.7^\circ}\right)^2 \\ &= 0.283. \end{aligned}$$

Since the two polarization components carry equal powers, the overall reflectance is simply the average of the isp and nsp values, that is, $R = 0.18$.

Solution 3.6 Circularly polarized light can be regarded as the sum of two linearly polarized waves, with a phase shift of $\pi/2$ between them, and with their polarizations in any pair of orthogonal directions in the plane perpendicular to the propagation direction. We can choose these directions to be the isp and nsp directions. For light incident at the Brewster angle, the reflectance is zero for the isp component, so the reflected wave is linearly polarized normal to the scattering plane.

Of course, this result is not restricted to circularly polarized incident light. Any beam reflected from a surface at the Brewster angle will be polarized normal to the scattering plane.

Solution 3.7 (a) The beam will be reflected back and forth in the plate if its angle of incidence to the interfaces is greater than the critical angle, θ_crit, which is given by

$$\sin\theta_\mathrm{crit} = n_2/n_1,$$

so that

$$\theta_\mathrm{crit} = \sin^{-1}(1.50/1.55) = 75.4^\circ.$$

So for angles of incidence between $75.4°$ and $90°$, the beam will be confined within the plate (almost!).

(b) For angles of incidence greater than the critical angle, the electric field is evanescent in the blocks either side of the plate, as discussed in Subsection 3.4.5 of Book 3. The field decays exponentially with distance z from the boundary, as $\exp[-z/\delta]$, where $\delta^{-1} = k_{\rm t}|\cos\theta_{\rm t}|$. The wavenumber in the blocks is given by

$$k_{\rm t} = \frac{2\pi n}{\lambda_0} = \frac{2\pi \times 1.50}{1500 \times 10^{-9}\,{\rm m}} = 6.28 \times 10^6\,{\rm m}^{-1}.$$

$$\begin{aligned}\cos\theta_{\rm t} &= \sqrt{(1 - \sin^2\theta_{\rm t})}\\ &= \sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_{\rm i}\right)^2}\\ &= \sqrt{1 - \left(\frac{1.55}{1.50}\sin 80°\right)^2}\\ &= \sqrt{-0.0356} = {\rm i}\,0.189.\end{aligned}$$

So the typical penetration depth δ for the field is

$$\begin{aligned}\delta &= (k_{\rm t}|\cos\theta_{\rm t}|)^{-1}\\ &= (6.28 \times 10^6\,{\rm m}^{-1} \times 0.189)^{-1}\\ &= 8.4 \times 10^{-7}\,{\rm m}.\end{aligned}$$

This is approximately half of the free space wavelength (1.5×10^{-6}m*).*

Book 3 Chapter 4

Solution 4.1 Since the wave is travelling in the $\mathbf{e}_y$-direction, the amplitude of the magnetic (and electric) field will decrease exponentially with the y-coordinate:

$$B_0 \propto \exp[-k_{\rm imag}\,y] = \exp\left[-\frac{n_{\rm imag}\omega}{c}\,y\right].$$

So the ratio of the amplitudes at the two points is

$$\frac{\exp\left[-\dfrac{n_{\rm imag}\omega}{c}\,y_2\right]}{\exp\left[-\dfrac{n_{\rm imag}\omega}{c}\,y_1\right]} = \exp\left[-\frac{n_{\rm imag}\omega}{c}\,(y_2 - y_1)\right].$$

Solution 4.2

$$k_{\rm real} = a_2, \quad k_{\rm imag} = a_1, \quad \omega = a_3,$$

$$n_{\rm real} = \frac{c}{\omega}\,k_{\rm real} = \frac{a_2 c}{a_3}, \quad n_{\rm imag} = \frac{c}{\omega}\,k_{\rm imag} = \frac{a_1 c}{a_3}.$$

Since $\varepsilon = n^2 = n_{\rm real}^2 - n_{\rm imag}^2 + 2{\rm i}\,n_{\rm real}n_{\rm imag}$, then

$$\begin{aligned}\varepsilon_{\rm real} &= n_{\rm real}^2 - n_{\rm imag}^2\\ &= \frac{(a_2^2 - a_1^2)c^2}{a_3^2},\end{aligned}$$

$$\begin{aligned}\varepsilon_{\rm imag} &= 2n_{\rm real}n_{\rm imag}\\ &= \frac{2a_1a_2c^2}{a_3^2},\end{aligned}$$

$$\begin{aligned}v_{\rm phase} &= c/n_{\rm real}\\ &= a_3/a_2,\end{aligned}$$

absorption length $= 1/k_{\rm imag} = 1/a_1$.

Solution 4.3 (a) (i) Refractive index $n = n_{\rm real} + {\rm i}\,n_{\rm imag} = \sqrt{\varepsilon}$, and when $\varepsilon_{\rm real} \gg \varepsilon_{\rm imag}$, then (as shown in Exercise 4.2 in Book 3)

$$n_{\rm real} = \sqrt{\varepsilon_{\rm real}} = \sqrt{1.46} = 1.2,$$

$$n_{\rm imag} = \frac{\varepsilon_{\rm imag}}{2\sqrt{\varepsilon_{\rm real}}} = \frac{1.0 \times 10^{-10}}{2 \times \sqrt{1.46}} = 4.1 \times 10^{-11}\,.$$

(ii) The wavenumber is complex and is given by

$$\begin{aligned}k &= k_{\rm real} + {\rm i}\,k_{\rm imag}\\ &= \frac{\omega}{c}(n_{\rm real} + {\rm i}\,n_{\rm imag})\\ &= \frac{2\pi}{\lambda_0}(n_{\rm real} + {\rm i}\,n_{\rm imag})\\ &= \frac{2\pi}{2.0 \times 10^{-6}\,{\rm m}}(1.21 + {\rm i}\,4.1 \times 10^{-11}\,)\\ &= 3.8 \times 10^6\,{\rm m}^{-1} + {\rm i}\,1.3 \times 10^{-4}\,{\rm m}^{-1}.\end{aligned}$$

(iii) The phase speed is real and is given by

$$\begin{aligned}v_{\rm phase} &= c/n_{\rm real}\\ &= 3.00 \times 10^8\,{\rm m\,s^{-1}}/1.2\\ &= 2.5 \times 10^8\,{\rm m\,s^{-1}}.\end{aligned}$$

(iv) The amplitude falls exponentially with distance, as $\exp[-k_{\rm imag}y]$, so over a distance of 1.0 km the amplitude falls by a factor

$$\exp\left[-1.3 \times 10^{-4}\,{\rm m}^{-1} \times 1.0 \times 10^3\,{\rm m}\right] = 0.88,$$

which corresponds to a decrease of about 12%.

(b) The physical electric field of a plane wave travelling in the x-direction and polarized in the y-direction, with amplitude E_0 at the origin is

$$\begin{aligned}\mathbf{E}_{\rm phys} &= {\rm Re}\big\{E_0\exp[{\rm i}(k_{\rm real}x + {\rm i}\,k_{\rm imag}x - \omega t)]\big\}\,\mathbf{e}_y\\ &= {\rm Re}\big\{E_0\exp[-k_{\rm imag}x]\,\exp[{\rm i}(k_{\rm real}x - \omega t)]\big\}\,\mathbf{e}_y\\ &= E_0\exp[-k_{\rm imag}x]\,\cos(k_{\rm real}x - \omega t)\,\mathbf{e}_y.\end{aligned}$$

Substituting the values of $k_{\rm real}$ and $k_{\rm imag}$ from part (a), and with $\omega = 2\pi c/\lambda_0 = 9.4 \times 10^{14}\,{\rm s}^{-1}$, we obtain

$$\begin{aligned}\mathbf{E}_{\rm phys} &= E_0\exp[-1.3 \times 10^{-4}\,{\rm m}^{-1}x]\times\\ &\cos(3.8 \times 10^6\,{\rm m}^{-1}x - 9.4 \times 10^{14}\,{\rm s}^{-1}t)\,\mathbf{e}_y.\end{aligned}$$

We determine the magnetic field using the same method as in Worked Example 4.2. Since $\mathbf{E}_{\rm phys} \times \mathbf{B}_{\rm phys}$ must be in the propagation direction, $\mathbf{e}_x$, we deduce that $\mathbf{B}_{\rm phys}$ must be in the $\mathbf{e}_z$-direction.

Also, the amplitude of the magnetic field is $(n/c)E_0$, so

$$\mathbf{B} = \frac{(n_{\text{real}} + \mathrm{i}\, n_{\text{imag}})}{c} E_0 \exp[-k_{\text{imag}} x] \times \exp[\mathrm{i}(k_{\text{real}} x - \omega t)]\, \mathbf{e}_z.$$

The physical magnetic field is the real part of this, that is

$$\mathbf{B}_{\text{phys}} = \frac{E_0}{c} \exp[-k_{\text{imag}} x] \times \big(n_{\text{real}} \cos\left(k_{\text{real}} x - \omega t\right) - n_{\text{imag}} \sin\left(k_{\text{real}} x - \omega t\right)\big)\, \mathbf{e}_z.$$

This can be rewritten as

$$\mathbf{B}_{\text{phys}} = \frac{E_0}{c} \exp[-k_{\text{imag}} x]\, \alpha \cos(k_{\text{real}} x - \omega t + \beta)\, \mathbf{e}_z,$$

where

$$\alpha = \sqrt{n_{\text{real}}^2 + n_{\text{imag}}^2} = n_{\text{real}}, \quad \text{since} \quad n_{\text{real}} \gg n_{\text{imag}},$$

and

$$\begin{aligned} \beta &= \tan^{-1}\left(n_{\text{imag}}/n_{\text{real}}\right) \\ &= \tan^{-1}\left(4.1 \times 10^{-11}/1.2\right) \\ &= \tan^{-1}\left(3.4 \times 10^{-11}\right), \end{aligned}$$

which is small enough to be negligible. Then since $n_{\text{real}}/c = 1/v_{\text{phase}}$, the magnetic field is

$$\begin{aligned} \mathbf{B}_{\text{phys}} &= \frac{E_0}{v_{\text{phase}}} \exp[-k_{\text{imag}} x] \cos(k_{\text{real}} x - \omega t)\, \mathbf{e}_z \\ &= \frac{E_0}{2.5 \times 10^8\,\mathrm{m\,s^{-1}}} \exp[-1.3 \times 10^{-4}\,\mathrm{m^{-1}}\, x] \times \\ &\quad \cos(3.8 \times 10^6\,\mathrm{m^{-1}}\, x - 9.4 \times 10^{14}\,\mathrm{s^{-1}}\, t)\, \mathbf{e}_z. \end{aligned}$$

Solution 4.4 The simple classical model predicts a complex dielectric function with real and imaginary parts given by Equations 4.8 and 4.9 of Book 3:

$$\varepsilon_{\text{real}}(\omega) = 1 + \omega_{\text{p}}^2 \frac{(\omega_{\text{n}}^2 - \omega^2)}{(\omega_{\text{n}}^2 - \omega^2)^2 + \omega^2\gamma^2},$$

$$\varepsilon_{\text{imag}}(\omega) = \omega_{\text{p}}^2 \frac{\omega\gamma}{(\omega_{\text{n}}^2 - \omega^2)^2 + \omega^2\gamma^2}.$$

(a) When $\omega \to 0$,

$$\varepsilon_{\text{real}} \to 1 + \frac{\omega_{\text{p}}^2}{\omega_{\text{n}}^2} = 1 + 2.00^2 = 5.0,$$

$$\varepsilon_{\text{imag}} \to 0.$$

(b) When $\omega \to \infty$,

$$\varepsilon_{\text{real}} \to 1 - \frac{\omega_{\text{p}}^2}{\omega^2} \to 1, \quad \text{and} \quad \varepsilon_{\text{imag}} \to \frac{\omega_{\text{p}}^2\gamma}{\omega^3} \to 0.$$

(c) $\varepsilon_{\text{real}} = 1$ when $\omega = \omega_{\text{n}} = 4.0 \times 10^{15}\,\mathrm{s^{-1}}$.

(d) At $\omega = \omega_{\text{n}} = 4.0 \times 10^{15}\,\mathrm{s^{-1}}$, $\varepsilon_{\text{imag}} = \omega_{\text{p}}^2/\omega_{\text{n}}\gamma = 2.00^2 = 4.0$.

Solution 4.5 (a) We ignore the possibly complicated geometry of the steak, and treat it as a semi-infinite slab. The microwave amplitude within the steak will then fall exponentially with distance z below the surface, as $\exp[-k_{\text{imag}} z]$, and the depth d at which the amplitude has dropped to half of the surface value is given by

$$\exp[-k_{\text{imag}} d] = \tfrac{1}{2}, \quad \text{or} \quad d = \ln 2/k_{\text{imag}}.$$

Now $k_{\text{imag}} = n_{\text{imag}}\omega/c$, and

$$\begin{aligned} n_{\text{imag}} &= \frac{-\varepsilon_{\text{real}} + \sqrt{\varepsilon_{\text{real}}^2 + \varepsilon_{\text{imag}}^2}}{2} \\ &= \frac{-40 + \sqrt{40^2 + 12^2}}{2} \\ &= 0.8806. \end{aligned}$$

So

$$\begin{aligned} d &= \frac{c \ln 2}{n_{\text{imag}}\omega} \\ &= \frac{3.00 \times 10^8\,\mathrm{m\,s^{-1}} \times \ln 2}{0.8806 \times 2\pi \times 2.45 \times 10^9\,\mathrm{Hz}} \\ &= 1.5\,\mathrm{cm}. \end{aligned}$$

(b) For polystyrene, $\varepsilon_{\text{imag}} \ll \varepsilon_{\text{real}}$ and we can use the approximation

$$\begin{aligned} n_{\text{imag}} &\simeq \frac{\varepsilon_{\text{imag}}}{2\sqrt{\varepsilon_{\text{real}}}} \\ &= \frac{3 \times 10^{-5}}{2\sqrt{1.03}} \\ &= 1.478 \times 10^{-5}. \end{aligned}$$

So

$$\begin{aligned} d &= \frac{c \ln 2}{n_{\text{imag}}\omega} \\ &= \frac{3.00 \times 10^8\,\mathrm{m\,s^{-1}} \times \ln 2}{1.478 \times 10^{-5} \times 2\pi \times 2.45 \times 10^9\,\mathrm{Hz}} \\ &= 9.1 \times 10^2\,\mathrm{m}. \end{aligned}$$

So the microwave amplitude will hardly be affected by even the thickest polystyrene containers that are used in a microwave oven. In other words, the microwaves pass through polystyrene with minimal absorption.

Solution 4.6 (a) From Snell's law, $\sin\theta_{\text{t}} = \sin\theta_{\text{i}}/n$, so because blue light is more strongly refracted than red, then $n_{\text{blue}} > n_{\text{red}}$. Therefore, since blue light has a higher frequency than red light, $\mathrm{d}n/\mathrm{d}\omega > 0$.

(b) The group speed is given by (Equation 4.27 of Book 3)

$$v_{\text{group}} = \frac{c}{\omega\, \mathrm{d}n/\mathrm{d}\omega + n},$$

so since $\mathrm{d}n/\mathrm{d}\omega$ is positive, $v_{\text{group}} < c/n$. But the phase speed is c/n, so for light in glass the group speed is less than the phase speed.

However, since $n \sim 1.5$ for glass, both speeds are much less than the speed of light in a vacuum.

(c) Since $v_{\text{group}} < v_{\text{phase}}$ for light in glass, the individual maxima and minima of the electric field will travel faster than the envelope of the pulse, so they will progress from the rear to the front of the pulse.

Solution 4.7 (a)

$$n_{\text{blue}} = 1.522 + \frac{4.59 \times 10^{-15}\,\text{m}^2}{(4 \times 10^{-7}\,\text{m})^2} = 1.551,$$

$$n_{\text{red}} = 1.522 + \frac{4.59 \times 10^{-15}\,\text{m}^2}{(6 \times 10^{-7}\,\text{m})^2} = 1.535.$$

(b) Phase speed is given by $v = c/n$, so

$$\begin{aligned} v_{\text{blue}} &= 2.998 \times 10^8\,\text{m}\,\text{s}^{-1}/1.551 \\ &= 1.933 \times 10^8\,\text{m}\,\text{s}^{-1}, \\ v_{\text{red}} &= 2.998 \times 10^8\,\text{m}\,\text{s}^{-1}/1.535 \\ &= 1.953 \times 10^8\,\text{m}\,\text{s}^{-1}. \end{aligned}$$

Note that we have used a more precise value for c because the refractive index data is given to four significant figures.

(c) The group speed for waves with frequencies around ω is given by Equation 4.27 of Book 3:

$$v_{\text{group}} = \frac{c}{\omega\,\mathrm{d}n/\mathrm{d}\omega + n}.$$

Now

$$n = A + \frac{B}{\lambda_0^2} = A + \frac{\omega^2 B}{4\pi^2 c^2},$$

and so

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} = \frac{2\omega B}{4\pi^2 c^2},$$

and

$$\begin{aligned} v_{\text{group}} &= \frac{c}{\omega \dfrac{2\omega B}{4\pi^2 c^2} + A + \dfrac{\omega^2 B}{4\pi^2 c^2}} \\ &= \frac{c}{A + \dfrac{3\omega^2 B}{4\pi^2 c^2}} \\ &= \frac{c}{A + \dfrac{3B}{\lambda_0^2}}. \end{aligned}$$

Hence

$$\begin{aligned} v_{\text{group blue}} &= \frac{2.998 \times 10^8\,\text{m}\,\text{s}^{-1}}{1.522 + \dfrac{3 \times 4.59 \times 10^{-15}\,\text{m}^2}{(4 \times 10^{-7}\,\text{m})^2}} \\ &= 1.864 \times 10^8\,\text{m}\,\text{s}^{-1}, \\ v_{\text{group red}} &= \frac{2.998 \times 10^8\,\text{m}\,\text{s}^{-1}}{1.522 + \dfrac{3 \times 4.59 \times 10^{-15}\,\text{m}^2}{(6 \times 10^{-7}\,\text{m})^2}} \\ &= 1.921 \times 10^8\,\text{m}\,\text{s}^{-1}. \end{aligned}$$

(d) The transmission angle is given by Snell's law, $n_1 \sin\theta_{\text{i}} = n_2 \sin\theta_{\text{t}}$. With $n_1 = 1.000$ for air, and $\theta_{\text{i}} = 30°$,

$$\begin{aligned} \theta_{\text{t blue}} &= \sin^{-1}(\sin 30°/n_{\text{blue}}) \\ &= \sin^{-1}(0.5/1.551) = 18.81°, \\ \theta_{\text{t red}} &= \sin^{-1}(\sin 30°/n_{\text{red}}) \\ &= \sin^{-1}(0.5/1.535) = 19.01°. \end{aligned}$$

So the blue light is refracted $0.2°$ more than the red light, a 2% difference in the angle of deviation $(\theta_{\text{i}} - \theta_{\text{t}})$.

Solution 4.8 The absorption length is the characteristic distance associated with the exponential decay of field amplitude in the factor $\exp\left[-k_{\text{imag}}\,z\right]$, that is,

$$\begin{aligned} \text{absorption length} &= \frac{1}{k_{\text{imag}}} \\ &= \frac{c}{n_{\text{imag}}\omega} = \frac{\lambda_0}{2\pi n_{\text{imag}}}. \end{aligned}$$

The values of n_{imag} for water and silica at the three frequencies can be estimated from Figures 4.15 and 4.17 of Book 3, and are tabulated below.

	500 nm	1000 nm	2000 nm
n_{imag} for water	10^{-8}	3×10^{-6}	2×10^{-3}
n_{imag} for silica	10^{-11}	10^{-12}	10^{-10}

The absorption lengths calculated from these data are as follows

	500 nm	1000 nm	2000 nm
absorpn length for water / m	8	5×10^{-2}	2×10^{-4}
absorpn length for silica / m	8×10^3	2×10^5	3×10^3

Note that the absorption length is three to seven orders of magnitude longer in silica than in water.

Solution 4.9 We require that the separation of the two wavelength components of each pulse is less than 10% of the $10\,\mu$s interpulse separation, that is, less than $1\,\mu$s. The time taken for the component centred on the free space wavelength λ_1 to travel distance D is

$$\frac{D}{v_{\text{group}}} = \frac{D}{c}\left(A + \frac{B}{\lambda_1^2}\right),$$

and the time for the component centred on λ_2 to travel the same distance is

$$\frac{D}{c}\left(A + \frac{B}{\lambda_2^2}\right).$$

The difference between these times is

$$\frac{D}{c}\left[\left(A+\frac{B}{\lambda_1^2}\right)-\left(A+\frac{B}{\lambda_2^2}\right)\right]$$
$$=\frac{D}{3.00\times10^8\,\mathrm{m\,s^{-1}}}\times\left[1.062\times10^{-14}\,\mathrm{m^2}\right]$$
$$\times\left[\frac{1}{(600\times10^{-9}\,\mathrm{m})^2}-\frac{1}{(650\times10^{-9}\,\mathrm{m})^2}\right]$$
$$=D\times1.455\times10^{-11}\,\mathrm{m^{-1}\,s}.$$

This difference must be less than $1\,\mu$s, so

$$D\times1.455\times10^{-11}\,\mathrm{m^{-1}\,s}<1\times10^{-6}\,\mathrm{s},\ \text{or}\ D<69\,\mathrm{km}.$$

Book 3 Chapter 5

Solution 5.1 (a) $\omega_\mathrm{p}=\sqrt{ne^2/m\varepsilon_0}$ is the plasma frequency, which is the frequency of collective oscillations of the free electrons in the conductor. τ_c is the collision time for the free electrons.

(b) $\varepsilon(\omega)$ is real when the imaginary term in the denominator is negligible, that is, when $\omega\gg1/\tau_\mathrm{c}$. In this frequency range, $\varepsilon(\omega)=1-\omega_\mathrm{p}^2/\omega^2$.

(c) For very high frequencies, that is, when $\omega\gg\omega_\mathrm{p}^2$, the relative permittivity function is unity. This corresponds to the electrons, permanent dipoles, etc, being unable to respond to the rapid changes of the electric field because of their inertia.

(d) In the range $1/\tau_\mathrm{c}\ll\omega<\omega_\mathrm{p}$, the relative permittivity is real (see part (b)), and it is negative. This means that the refractive index ($n=\sqrt{\varepsilon}$) and the wavenumber are imaginary, which means that electromagnetic disturbances are evanescent, rather than travelling waves.

(e) For $\omega\ll1/\tau_\mathrm{c}$, we obtain

$$\varepsilon(\omega)=1-\frac{\omega_\mathrm{p}^2}{\mathrm{i}\,\omega/\tau_\mathrm{c}}=1+\frac{\mathrm{i}\,\omega_\mathrm{p}^2\tau_\mathrm{c}}{\omega}.$$

Now $\omega_\mathrm{p}^2=ne^2/m\varepsilon_0$, and $\sigma=ne^2\tau_\mathrm{c}/m$, so in the low frequency limit,

$$\varepsilon(\omega)=1+\frac{\mathrm{i}\,\sigma}{\omega\varepsilon_0}.$$

Solution 5.2 When a wave meets the surface of a dielectric, it is partly reflected and partly transmitted into the material. The transmitted wave has wavenumber $k=k_\mathrm{real}+\mathrm{i}\,k_\mathrm{imag}=(n_\mathrm{real}+\mathrm{i}\,n_\mathrm{imag})\,\omega/c$, it travels at speed $v=c/n_\mathrm{real}$, and its amplitude decays exponentially with distance as $\exp[-k_\mathrm{imag}z]$. For a material like glass the reflection is small, and the attenuation of the wave is also small. The electric and magnetic fields of the transmitted plane waves are transverse to the direction of propagation, orthogonal to each other, and they are in phase (as long as the attenuation is small).

At the surface of a conductor, the wave is almost completely reflected. The electric field penetrates a very small distance into the conductor, but the disturbance is evanescent, dying away exponentially within the skin depth, which is much smaller than the free space wavelength of the radiation. A small amount of energy is dissipated in the surface layer by the currents set up by the oscillating electric field. As with dielectrics, the electric and magnetic fields are transverse to the direction of propagation and orthogonal to each other, but they are out of phase by $\pi/4$ for a good conductor.

Solution 5.3 The electric field, and therefore the current density, decays exponentially with depth inside the conductor:

$$J(x)=\sigma E(x)=\sigma E_0\exp[-x/\delta],$$

where σ is the conductivity and the skin depth is $\delta=\sqrt{2/\mu_0\sigma\omega}$. Since $a\gg t$, we ignore the field penetration at the edges, and treat this as a one-dimensional, planar problem, with $J(x)$ constant across an element of area of width a and thickness δx at distance x below the surface. The total current flowing across a plane perpendicular to the length is

$$I=2\int_0^\infty\sigma E_0\exp[-x/\delta]\,a\,\mathrm{d}x,$$

where the factor of 2 takes account of the penetration of field from both surfaces of the strip, and the upper limit of the integral can be taken as infinity as long as the thickness t is much greater than the skin depth so that the field is essentially zero in the middle. Thus

$$I=\left[-2\delta a\sigma E_0\exp[-x/\delta]\right]_0^\infty=2\delta a\sigma E_0.$$

Now the voltage difference between the ends of the strip is given by $V=E_0L$, so its resistance is

$$R(f)=\frac{V}{I}=\frac{E_0L}{2a\sigma E_0\,\delta(f)}=\frac{L}{2a\sigma\,\delta(f)}.$$

The DC resistance is given by

$$R(0)=\frac{\text{length}}{\text{area}\times\sigma}=\frac{L}{at\sigma},$$

so

$$\frac{R(f)}{R(0)}=\frac{t}{2\,\delta(f)}=\frac{t}{2}\sqrt{\frac{\mu_0\sigma\omega}{2}}=\frac{t}{2}\sqrt{\pi\mu_0\sigma f}.$$

The resistance is equivalent to that of a strip that is two skin depths thick, because of field penetration from the two surfaces.

Solution 5.4 The cut-off frequency for the m th mode, which has m half-wavelengths of electric field variation fitted between the plates, is given by

$$f_\mathrm{c}=\frac{c}{\lambda_\mathrm{c}}=\frac{m}{2a}c=m\frac{3.00\times10^8\,\mathrm{m\,s^{-1}}}{2\times0.10\,\mathrm{m}}=1.5m\,\mathrm{GHz}.$$

The $m=6$ mode has $f_\mathrm{c}=9.0\,\mathrm{GHz}$, but the $m=7$ mode has $f_\mathrm{c}=10.5\,\mathrm{GHz}$, so six TE modes can propagate at $10\,\mathrm{GHz}$.

Solution 5.5 The magnetic field will have the same time dependence, $\exp[-\mathrm{i}\,\omega t]$, as the electric field, so we can write Faraday's law as

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \mathrm{i}\,\omega\mathbf{B}.$$

The electric field has only an x-component, so its curl reduces to

$$\operatorname{curl}\mathbf{E} = \frac{\partial E_x}{\partial z}\,\mathbf{e}_y - \frac{\partial E_x}{\partial y}\,\mathbf{e}_z = \mathrm{i}\,\omega\mathbf{B}.$$

Equating vector components on left and right,

$$\begin{aligned}
B_x =&0,\\
B_y =&\frac{1}{\mathrm{i}\,\omega}\frac{\partial E_x}{\partial z} = \frac{1}{\mathrm{i}\,\omega}\,k_0\cos\theta\,E_0\,2\mathrm{i}\times\\
&\quad\cos(k_0 z\cos\theta)\exp[\mathrm{i}(k_0 y\sin\theta - \omega t)]\\
=&E_0\,\frac{2k_0\cos\theta}{\omega}\,\cos(k_0 z\cos\theta)\times\\
&\quad\exp[\mathrm{i}(k_0 y\sin\theta - \omega t)],\\
B_z =&-\frac{1}{\mathrm{i}\,\omega}\frac{\partial E_x}{\partial y} = -\frac{1}{\mathrm{i}\,\omega}\,\mathrm{i}\,k_0\sin\theta\,E_0\,2\mathrm{i}\times\\
&\quad\sin(k_0 z\cos\theta)\exp[\mathrm{i}(k_0 y\sin\theta - \omega t)]\\
=&-\mathrm{i}\,E_0\,\frac{2k_0\sin\theta}{\omega}\,\sin(k_0 z\cos\theta)\times\\
&\quad\exp[\mathrm{i}(k_0 y\sin\theta - \omega t)].
\end{aligned}$$

Note that $\mathbf{B}$ has a component in the y-direction — the direction of propagation — so the magnetic field is not transverse.

Solution 5.6 The dispersion relation is

$$k_{\mathrm{gw}} = \left[\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{1/2},$$

and this can be rewritten as

$$\omega = c\left[k_{\mathrm{gw}}^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}.$$

Then

$$\begin{aligned}
v_{\mathrm{group}} &= \frac{\mathrm{d}\omega}{\mathrm{d}k_{\mathrm{gw}}}\\
&= \tfrac{1}{2}\times 2ck_{\mathrm{gw}}\left[k_{\mathrm{gw}}^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-1/2}\\
&= \frac{c^2 k_{\mathrm{gw}}}{\omega}.
\end{aligned}$$

This can be written in the alternative forms

$$\begin{aligned}
v_{\mathrm{group}} &= \frac{c^2}{\omega}\left[\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{1/2}\\
&= c\left[1 - \left(\frac{m\lambda_0}{2a}\right)^2 - \left(\frac{n\lambda}{2b}\right)^2\right]^{1/2}.
\end{aligned}$$

Note that since $v_{\mathrm{phase}} = \omega/k_{\mathrm{gw}}$, the phase and group speeds are related by $v_{\mathrm{phase}}v_{\mathrm{group}} = c^2$.

Solution 5.7 (a) The cut-off frequency for the TE_{mn} mode is given by Equation 5.51 of Book 3:

$$f_{mn} = c\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2},$$

where a and b are the longer and shorter dimensions of the waveguide. In this case, $b = a/2$ and so

$$\begin{aligned}
f_{mn} &= \frac{c}{2a}\sqrt{m^2 + 4n^2}\\
&= \frac{3.00\times 10^8\,\mathrm{m\,s^{-1}}}{2\times 0.034\,\mathrm{m}}\sqrt{m^2+4n^2}\\
&= 4.411\sqrt{m^2+4n^2}\,\mathrm{GHz}.
\end{aligned}$$

The lowest four cut-off frequencies will be those for the TE_{10}, TE_{20}, TE_{01} and TE_{11} modes, for which the values of $\sqrt{m^2+4n^2}$ are 1, 2, 2 and $\sqrt{5}$, respectively, leading to cut-off frequencies of $4.4\,\mathrm{GHz}$, $8.8\,\mathrm{GHz}$, $8.8\,\mathrm{GHz}$ and $9.9\,\mathrm{GHz}$, respectively.

(b) The phase speed for the TE_{mn} mode is given by

$$\begin{aligned}
v_{\mathrm{phase}} &= \frac{\omega}{k_{\mathrm{gw}}}\\
&= \omega\Big/\left(\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right)^{1/2}\\
&= c\Big/\left(1 - \frac{c^2}{\omega^2}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\right)^{1/2}\\
&= c\Big/\left(1 - \frac{c^2}{f^2}\left[\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2\right]\right)^{1/2}\\
&= c\left(1 - \frac{f_{mn}^2}{f^2}\right)^{-1/2}.
\end{aligned}$$

Substituting the values of the cut-off frequencies determined in part (a), the phase speeds for the four modes are calculated to be

mode	TE_{10}	TE_{20}	TE_{01}	TE_{11}
$v_{\mathrm{phase}}/10^8\,\mathrm{m\,s^{-1}}$	3.3	6.4	6.4	18.

The group speed is most easily determined using the result in Equation 5.45,

$$v_{\mathrm{group}} = c^2/v_{\mathrm{phase}},$$

which leads to

mode	TE_{10}	TE_{20}	TE_{01}	TE_{11}
$v_{\mathrm{group}}/10^8\,\mathrm{m\,s^{-1}}$	2.7	1.4	1.4	0.49.

Book 3 Chapter 6

Solution 6.1 For a collisionless plasma, the relative permittivity function is given by

$$\varepsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2},$$

where the plasma frequency is given by

$$\begin{aligned}
\omega_{\mathrm{p}} &= \sqrt{\frac{n_{\mathrm{e}}e^2}{m\varepsilon_0}} \\
&= \sqrt{\frac{10^{10}\,\mathrm{m}^{-3} \times (1.60 \times 10^{-19}\,\mathrm{C})^2}{9.11 \times 10^{-31}\,\mathrm{kg} \times 8.85 \times 10^{-12}\,\mathrm{F\,m}^{-1}}} \\
&= 5.635 \times 10^6\,\mathrm{s}^{-1}.
\end{aligned}$$

The phase speed is

$$v_{\mathrm{phase}} = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon(\omega)}} = \frac{c\omega}{\sqrt{\omega^2 - \omega_{\mathrm{p}}^2}},$$

and (as shown in Worked Example 6.1 of Book 3) the group speed is

$$\begin{aligned}
v_{\mathrm{group}} &= \frac{\mathrm{d}\omega}{\mathrm{d}k} = \left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1} \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\sqrt{\varepsilon(\omega)}\,\frac{\omega}{c}\right)\right)^{-1} \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\sqrt{\omega^2 - \omega_{\mathrm{p}}^2}}{c}\right)\right)^{-1} \\
&= \left(\frac{\omega}{c}\,\frac{1}{\sqrt{\omega^2 - \omega_{\mathrm{p}}^2}}\right)^{-1} \\
&= \frac{c}{\omega}\sqrt{\omega^2 - \omega_{\mathrm{p}}^2} \\
&= \frac{c^2}{v_{\mathrm{phase}}}.
\end{aligned}$$

Thus for a frequency of 1.00×10^6 Hz,

$$\begin{aligned}
v_{\mathrm{phase}} &= \frac{c\omega}{\sqrt{\omega^2 - \omega_{\mathrm{p}}^2}} \\
&= \frac{c}{\sqrt{1 - \omega_{\mathrm{p}}^2/\omega^2}} \\
&= \frac{3.00 \times 10^8\,\mathrm{m\,s}^{-1}}{\sqrt{1 - (5.64 \times 10^6\,\mathrm{s}^{-1}/2\pi \times 1.00 \times 10^6\,\mathrm{s}^{-1})^2}} \\
&= 6.8 \times 10^8\,\mathrm{m\,s}^{-1},
\end{aligned}$$

and

$$v_{\mathrm{group}} = \frac{c^2}{v_{\mathrm{phase}}} = 1.3 \times 10^8\,\mathrm{m\,s}^{-1}.$$

The speeds for frequencies of 2.00 MHz and 100 MHz are calculated in a similar way, and the values are tabulated below.

	1.00 MHz	2.00 MHz	100 MHz
$v_{\mathrm{phase}}\,/10^8\,\mathrm{m\,s}^{-1}$	6.8	3.4	3.0
$v_{\mathrm{group}}\,/10^8\,\mathrm{m\,s}^{-1}$	1.3	2.7	3.0

Solution 6.2 The interstellar medium is a very dilute plasma, so it can be regarded as collisionless, and we will ignore any thermal effects, and assume that there is no significant magnetic field, so that the plasma can be regarded as isotropic. Then the relative permittivity function is given by

$$\varepsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2},$$

where the plasma frequency in the interstellar medium is given by

$$\begin{aligned}
\omega_{\mathrm{p}} &= \sqrt{\frac{n_{\mathrm{e}}e^2}{m\varepsilon_0}} \\
&= \sqrt{\frac{3 \times 10^4\,\mathrm{m}^{-3} \times (1.60 \times 10^{-19}\,\mathrm{C})^2}{9.11 \times 10^{-31}\,\mathrm{kg} \times 8.85 \times 10^{-12}\,\mathrm{F\,m}^{-1}}} \\
&= 9.76 \times 10^3\,\mathrm{s}^{-1}.
\end{aligned}$$

As shown in Worked Example 6.1 of Book 3, the group speed for a material with this relative permittivity function is

$$v_{\mathrm{group}} = c\sqrt{1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}},$$

and since $\omega \gg \omega_{\mathrm{p}}$ for 100 MHz radiowaves and light, this can be approximated by

$$v_{\mathrm{group}} = c\left(1 - \tfrac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega^2}\right).$$

The difference between the arrival times for the radiowave and red pulses is

$$\begin{aligned}
\Delta t &= \frac{D}{v_{\mathrm{group\,radio}}} - \frac{D}{v_{\mathrm{group\,red}}} \\
&= \frac{D}{c}\left[\left(1 - \tfrac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{radio}}^2}\right)^{-1} - \left(1 - \tfrac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{red}}^2}\right)^{-1}\right] \\
&\simeq \frac{D}{c}\left[\left(1 + \tfrac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{radio}}^2}\right) - \left(1 + \tfrac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{red}}^2}\right)\right] \\
&= \frac{D\omega_{\mathrm{p}}^2}{2c}\left[\frac{1}{\omega_{\mathrm{radio}}^2} - \frac{1}{\omega_{\mathrm{red}}^2}\right].
\end{aligned}$$

Since $\omega_{\mathrm{radio}} \ll \omega_{\mathrm{red}}$, we can ignore the second term in brackets, so

$$\Delta t = \frac{10^{19}\,\mathrm{m} \times (9.76 \times 10^3\,\mathrm{s}^{-1})^2}{2 \times 3.00 \times 10^8\,\mathrm{m\,s}^{-1}(2\pi \times 10^8\,\mathrm{Hz})^2} = 4.0\,\mathrm{s}.$$

Solution 6.3 The relative permittivity of a collisionless isotropic plasma is given by

$$\varepsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}.$$

When $\omega < \omega_{\mathrm{p}}$, this is negative, which means that the refractive index $n\ (= \sqrt{\varepsilon})$ and the wavenumber $k\ (= n\omega/c)$ are imaginary. The electric field is therefore evanescent within the plasma. Neglecting

any spreading of the beam, the amplitude of the electric field will fall off as

$$\exp[\mathrm{i}kz] = \exp\left[\mathrm{i}\frac{\sqrt{\varepsilon}\,\omega}{c}\,z\right] = \exp\left[\mathrm{i}\frac{\sqrt{\omega^2-\omega_\mathrm{p}^2}}{c}\,z\right]$$
$$= \exp\left[-\frac{\sqrt{\omega_\mathrm{p}^2-\omega^2}}{c}\,z\right]$$
$$= \exp\left[-\frac{\omega_\mathrm{p}\sqrt{1-\omega^2/\omega_\mathrm{p}^2}}{c}\,z\right].$$

The value of this expression just inside the window, at $z = 0$, is 1, so the ratio of the amplitude of the electric field in the plasma at a distance z from the window to the amplitude just inside the window is just

$$\exp\left[-\frac{\omega_\mathrm{p}\sqrt{1-\omega^2/\omega_\mathrm{p}^2}}{c}\,z\right].$$

For $z = 0.5\,\mathrm{m}$ and $\omega/\omega_\mathrm{p} = 0.99$, this ratio is

$$\exp\left[-\frac{2\pi\times 10^8\,\mathrm{s^{-1}}\sqrt{1-0.99^2}}{3.00\times 10^8\,\mathrm{m\,s^{-1}}}\,0.5\,\mathrm{m}\right]$$
$$= 0.86.$$

For $\omega/\omega_\mathrm{p} = 0.90$ the ratio is 0.63, and for $\omega/\omega_\mathrm{p} = 0.50$ it is 0.40.

The limiting value of this ratio when $\omega/\omega_\mathrm{p} \ll 1$ is $\exp[-\pi/3] = 0.35$.

Solution 6.4 (a) Since $\omega_\mathrm{p} = \sqrt{n_\mathrm{e}e^2/m\varepsilon_0}$, the electron number density is

$$n_\mathrm{e} = \frac{\omega_\mathrm{p}^2 m\varepsilon_0}{e^2} =$$
$$\frac{(8.0\times 10^{10}\,\mathrm{s^{-1}})^2\times 9.11\times 10^{-31}\,\mathrm{kg}\times 8.85\times 10^{-12}\,\mathrm{F\,m^{-1}}}{(1.60\times 10^{-19}\,\mathrm{C})^2}$$
$$= 2.0\times 10^{18}\,\mathrm{m^{-3}}.$$

Also, since $\omega_\mathrm{c} = eB/m$, the magnetic field strength is

$$B = \frac{m\omega_\mathrm{c}}{e}$$
$$= \frac{9.11\times 10^{-31}\,\mathrm{kg}\times 3.0\times 10^8\,\mathrm{s^{-1}}}{1.60\times 10^{-19}\,\mathrm{C}}$$
$$= 1.7\times 10^{-3}\,\mathrm{T}.$$

(b) The radiowave frequency is much less than the plasma frequency and much less than the cyclotron frequency. This means that the relative permittivities for the RH and LH circularly polarized modes in the anisotropic plasma are

$$\varepsilon^\mathrm{RH} = \frac{\omega_\mathrm{p}^2}{\omega\omega_\mathrm{c}} \quad \text{and} \quad \varepsilon^\mathrm{LH} = -\frac{\omega_\mathrm{p}^2}{\omega\omega_\mathrm{c}},$$

as shown in Subsection 6.3.2 of Book 3. The LH mode is evanescent, so it is low frequency RH circularly polarized waves that are produced in the plasma. Their dispersion relation follows from

$$k^\mathrm{RH} = n^\mathrm{RH}\frac{\omega}{c} = \sqrt{\varepsilon^\mathrm{RH}}\frac{\omega}{c} = \frac{\omega_\mathrm{p}}{c}\sqrt{\frac{\omega}{\omega_\mathrm{c}}}.$$

Then

$$v_\mathrm{group}^\mathrm{RH} = \left(\frac{\mathrm{d}k^\mathrm{RH}}{\mathrm{d}\omega}\right)^{-1}$$
$$= \left(\frac{\omega_\mathrm{p}}{2c}\sqrt{\frac{1}{\omega\omega_\mathrm{c}}}\right)^{-1}$$
$$= 2c\frac{\sqrt{\omega\omega_\mathrm{c}}}{\omega_\mathrm{p}}.$$

Substituting the data provided,

$$v_\mathrm{group}^\mathrm{RH} = 2\times 3.00\times 10^8\,\mathrm{m\,s^{-1}}\times \frac{\sqrt{2\pi\times 13.56\times 10^6\,\mathrm{Hz}\times 3.0\times 10^8\,\mathrm{s^{-1}}}}{8.0\times 10^{10}\,\mathrm{s^{-1}}}$$
$$= 1.2\times 10^6\,\mathrm{m\,s^{-1}}.$$

Solution 6.5 (a) The microwave frequency ω_micro must equal the electron cyclotron frequency ω_c for resonance to occur. Thus

$$\omega_\mathrm{c} = \frac{eB}{m} = \omega_\mathrm{micro} = \frac{2\pi c}{\lambda_\mathrm{micro}},$$

and so

$$B = \frac{2\pi cm}{e\lambda_\mathrm{micro}}$$
$$= \frac{2\pi\times 3.00\times 10^8\,\mathrm{m\,s^{-1}}\times 9.11\times 10^{-31}\,\mathrm{kg}}{1.60\times 10^{-19}\,\mathrm{C}\times 3.0\times 10^{-2}\,\mathrm{m}}$$
$$= 0.36\,\mathrm{T}.$$

(b) For efficient heating we require that the collision frequency, $1/\tau_\mathrm{c}$, is much less than the cyclotron frequency, so that the microwave electric field can accelerate the electrons over a number of cycles before the electrons are scattered. From Figure 6.7 of Book 3 the height in the atmosphere where the pressure is 10^4 Pa is about 18 km, and from Figure 6.9 the collision frequency at this height (and pressure) is about $10^{11}\,\mathrm{s^{-1}}$. This is *greater* than ω_c ($\simeq 2\pi\times 10^{10}\,\mathrm{s^{-1}}$), so the heating would not be very efficient.

Book 3 Chapter 7

Solution 7.1 (a) Increasing the fibril diameter will increase the scattering cross-section, since there is more material to polarize and radiate energy.

(b) Increasing the refractive index of the fibril will increase the scattering cross-section, since this change increases the relative permittivity and therefore increases the polarization.

(c) Increasing the refractive index of the matrix will reduce the cross-section, because it reduces the difference between the polarization of the fibril and the matrix. If the fibrils and matrix had the same refractive index there would be no scattering.

(d) Increasing the spacing of the fibrils would not affect the cross-section of a single fibril. The fibrils are weak scatterers and are not affected by their neighbours.

(e) Increasing the wavelength of the radiation would reduce the scattering cross-section. The scattering will increase with frequency, just as Rayleigh scattering from molecules increases with frequency.

(f) Changing the polarization of the incident radiation from parallel to the fibril to perpendicular to the fibril will reduce the scattering cross-section. When the polarization is perpendicular to the fibril, there is no scattering in the polarization direction, and scattering from the different elements of the fibril will also cancel in the direction parallel to the fibril. This leaves scattered radiation concentrated around the forward and reverse directions.

Solution 7.2 The fundamental resonance corresponds to $\frac{1}{4}\lambda$ of the radiation in the length L of the hairpin, that is, $\lambda = 4L$. There is a node of the electric field at the short-circuited end and an antinode at the open end. Other resonances occur, each with the same boundary conditions at the ends of the hairpin. These all have an odd number of quarter-wavelengths fitted into the length of the hairpin, so $3\lambda/4 = L,\ 5\lambda/4 = L,\ 7\lambda/4 = L$, etc., or $\lambda = 4L/m$, where $m = 3, 5, 7$, etc. The frequencies of the resonances are given by $f = c/\lambda = mc/4L = mf_0$, so other resonance might be observed at $1.5\,\mathrm{GHz}$, $2.5\,\mathrm{GHz}$, $3.5\,\mathrm{GHz}$, etc.

Solution 7.3 (a) The fundamental resonance frequency f_{res} in a material in which the speed of electromagnetic radiation is v, the refractive index is n and the relative permittivity is ε is given by

$$f_{\mathrm{res}} = \frac{v}{\lambda} = \frac{c}{n\lambda} = \frac{c}{\sqrt{\varepsilon}\,\lambda} = \frac{c}{4\sqrt{\varepsilon}\,L} = \frac{f_0}{\sqrt{\varepsilon}}.$$

So in water, with $\varepsilon = 80$, $f_{\mathrm{res}} = 100\,\mathrm{MHz}/\sqrt{80} = 11\,\mathrm{MHz}$.

(b) In ethanol, with $\varepsilon = 25$, $f_{\mathrm{res}} = 100\,\mathrm{MHz}/\sqrt{25} = 20\,\mathrm{MHz}$.

(c) The relative permittivity of a cold collisionless plasma is given by

$$\varepsilon_{\mathrm{eff}} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} = 1 - \frac{f_{\mathrm{p}}^2}{f^2},$$

where the plasma frequency is

$$f_{\mathrm{p}} = \frac{\omega_{\mathrm{p}}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{n_{\mathrm{e}}e^2}{m\varepsilon_0}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2.0 \times 10^{13}\,\mathrm{m}^{-3} \times (1.60 \times 10^{-19}\,\mathrm{C})^2}{9.11 \times 10^{-31}\,\mathrm{kg} \times 8.85 \times 10^{-12}\,\mathrm{F\,m}^{-1}}}$$

$$= 40.11\,\mathrm{MHz}.$$

Thus

$$\varepsilon = 1 - \frac{40.11^2}{100^2} = 0.8391,$$

$$f_{\mathrm{res}} = 100\,\mathrm{MHz}/\sqrt{0.8391} = 110\,\mathrm{MHz}.$$

Solution 7.4 The fundamental resonance frequency in vacuum is

$$f_0 = \frac{c}{4L} = \frac{3.00 \times 10^8\,\mathrm{m\,s}^{-1}}{4 \times 1.00 \times 10^{-2}\,\mathrm{m}} = 7.5 \times 10^9\,\mathrm{Hz}.$$

In a plasma, the fundamental resonance frequency is given by

$$f_{\mathrm{res,\,p}} = \frac{f_0}{\sqrt{\varepsilon_{\mathrm{eff}}}} = \frac{f_0}{\sqrt{1 - f_{\mathrm{p}}^2/f_{\mathrm{res,\,p}}^2}},$$

where the plasma frequency is $f_{\mathrm{p}} = (1/2\pi)\sqrt{n_{\mathrm{e}}e^2/m\varepsilon_0}$. This simplifies to

$$f_{\mathrm{res,\,p}}^2 = f_{\mathrm{p}}^2 + f_0^2.$$

In this problem $f_{\mathrm{res,\,p}} = 2f_0$, so $f_{\mathrm{p}}^2 = 3f_0^2$. Hence

$$\frac{1}{4\pi^2}\frac{n_{\mathrm{e}}e^2}{m\varepsilon_0} = 3f_0^2,$$

which can be rearranged to give

$$n_{\mathrm{e}} = \frac{12\pi^2 m\varepsilon_0 f_0^2}{e^2}$$

$$= \frac{12\pi^2 \times 9.11 \times 10^{-31}\,\mathrm{kg}}{(1.60 \times 10^{-19}\,\mathrm{C})^2} \times 8.85 \times 10^{-12}\,\mathrm{F\,m}^{-1} \times (7.5 \times 10^9\,\mathrm{Hz})^2$$

$$= 2.1 \times 10^{18}\,\mathrm{m}^{-3}.$$

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